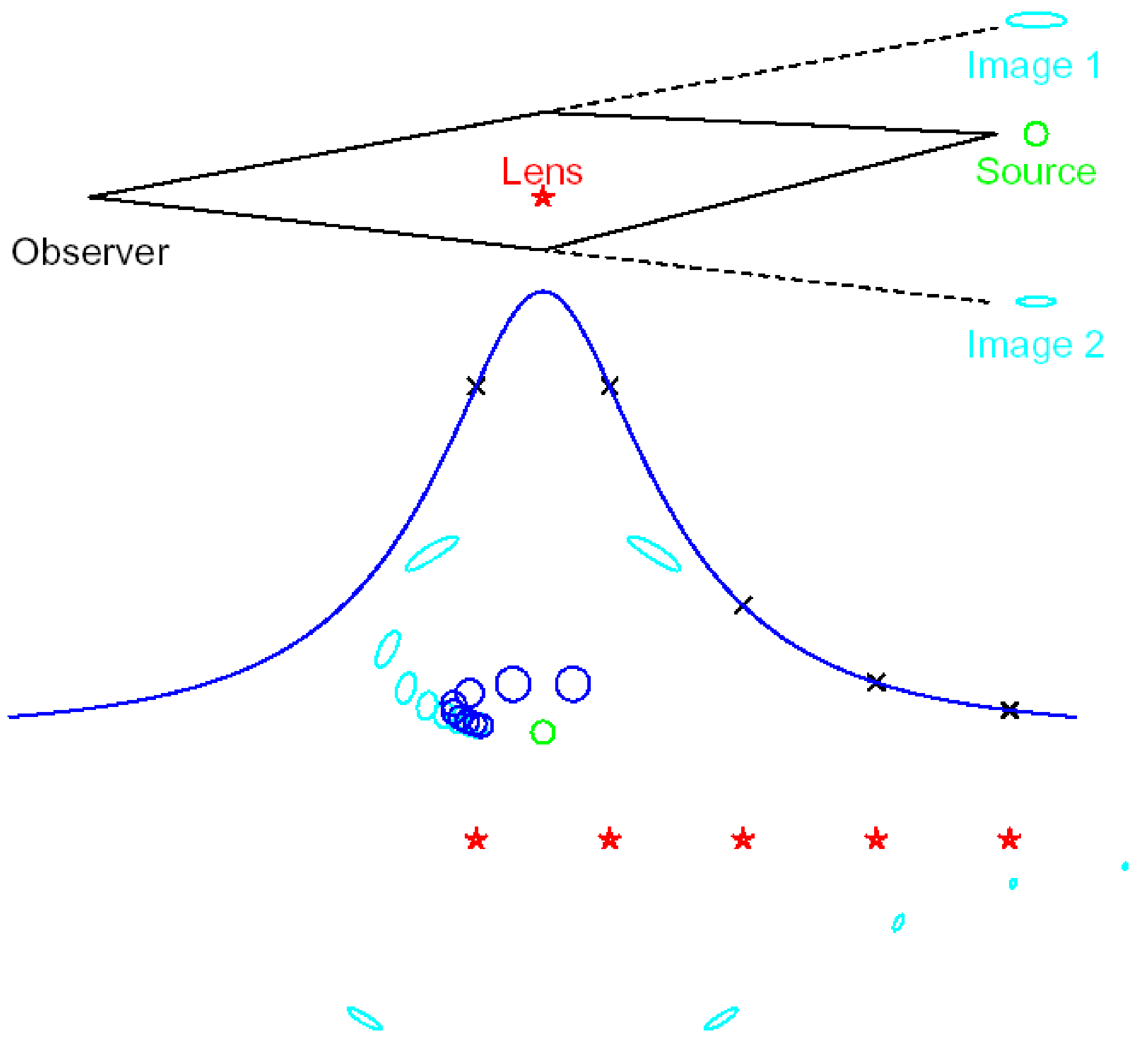
Exoplanet Microlensing:

Simple Lenses to Physics of Planets

Andy Gould (Ohio State)



Generation -1: Einstein (1912)

[Renn, Sauer, Stachel 1997, Science 275, 184]

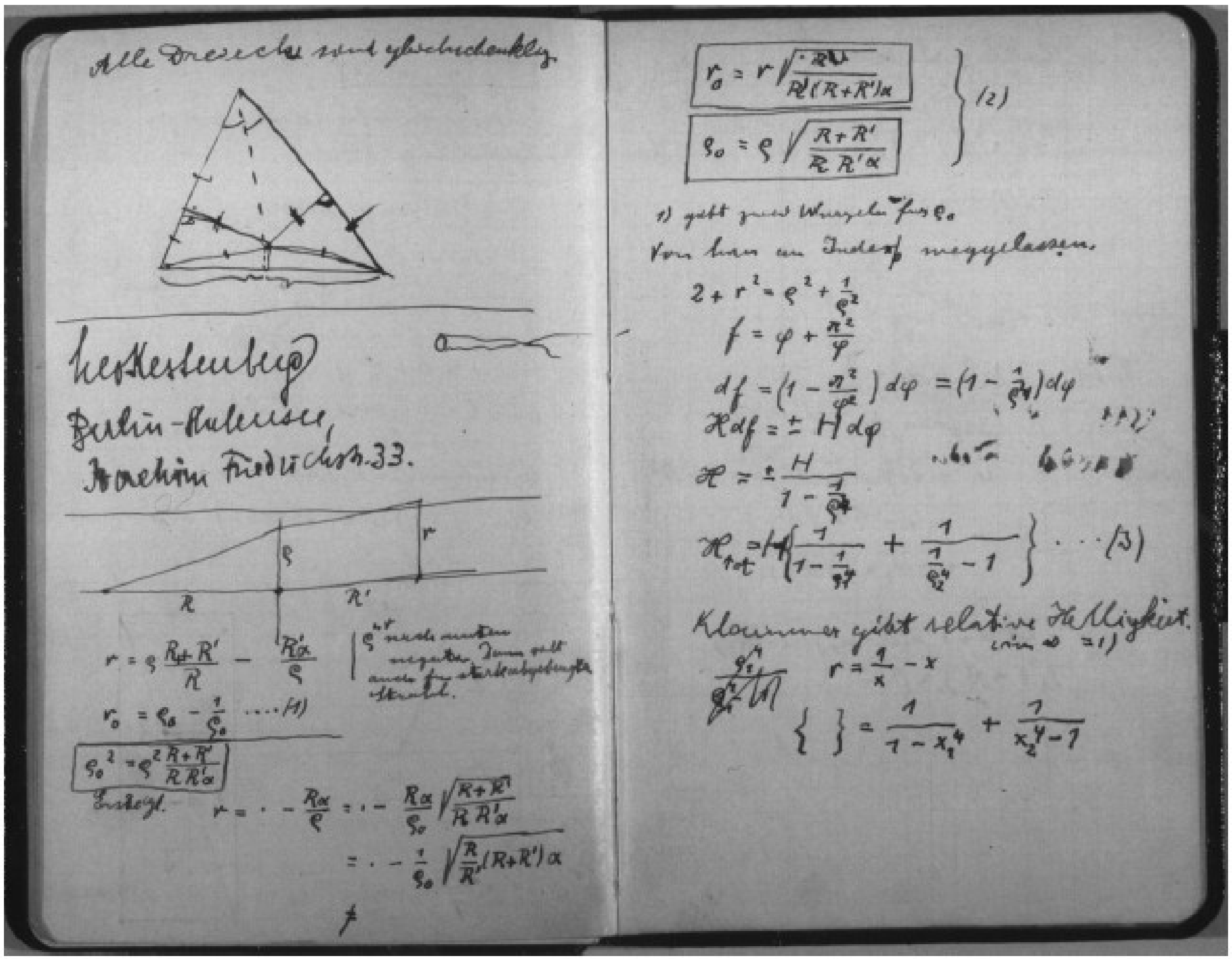
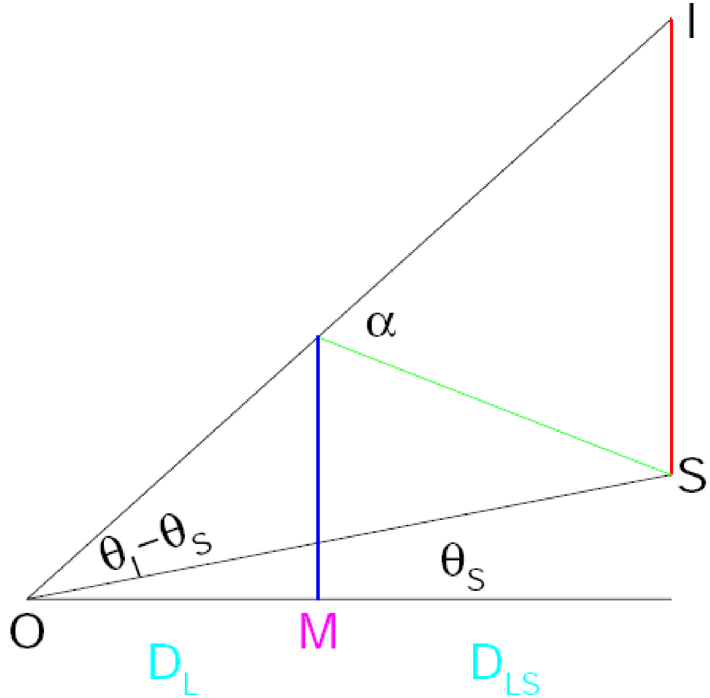
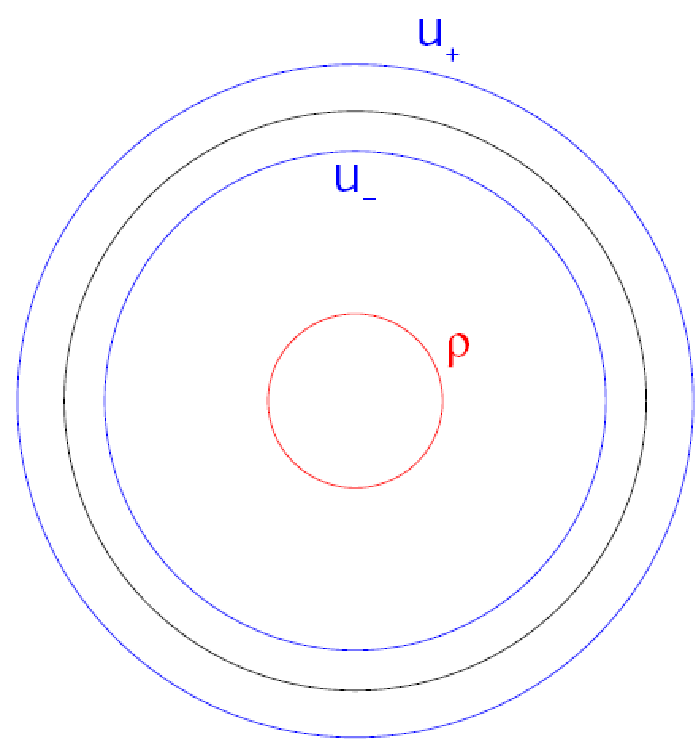


Fig. 1. Notes about gravitational lensing dated to 1912 on two pages of Einstein's scratch notebook (*12*). [Reproduced with permission of the Einstein Archives, Jewish National and University Library, Hebrew University of Jerusalem]



$$(\theta_I - \theta_S) D_S = \alpha D_{LS}$$

$$\alpha = 4GM/(D_L \theta_I c^2)$$



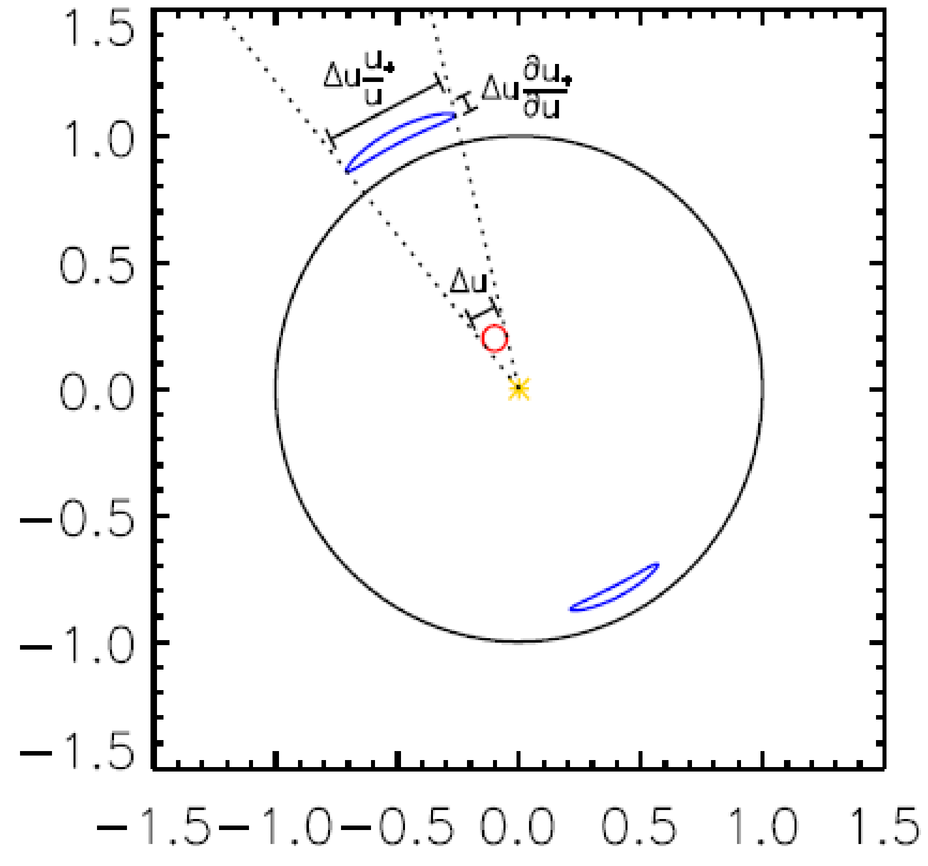
$$(\theta_I - \theta_S)\theta_I = \theta_E^{\,2} = (4GM/c^2)(D_{LS}/D_L D_S)$$

$$\theta_I/\theta_E = [u +/- (u^2 + 4)^{1/2}]/2; \quad u=\theta_S/\theta_E$$

Point-Lens Magnification

$$A = \frac{u^2+2}{u\sqrt{u^2+4}}$$

$$\begin{aligned}
u_\pm &= \frac{\sqrt{u^2+4}}{2} \pm u \\
A_\pm &= \pm \frac{u_\pm}{u}\frac{\partial u_\pm}{\partial u} \\
&= \pm \frac{1}{2}\frac{\partial u_\pm^2}{\partial u^2} \\
A_+ - A_- &= \frac{1}{2}\left(\frac{\partial u_+^2}{\partial u^2} + \frac{\partial u_-^2}{\partial u^2}\right) = \frac{\partial(u^2+2)}{\partial u^2} = 1 \\
A_\pm &= \frac{A \pm 1}{2}
\end{aligned}$$



$$A = A_+ + A_- = \frac{1}{2}\left(\frac{\partial u_+^2}{\partial u^2} - \frac{\partial u_-^2}{\partial u^2}\right) = \frac{\partial(u\sqrt{u^2+4})}{2u\partial u} = \frac{u^2+2}{u\sqrt{u^2+4}}$$

Point-Lens Limiting Formulae

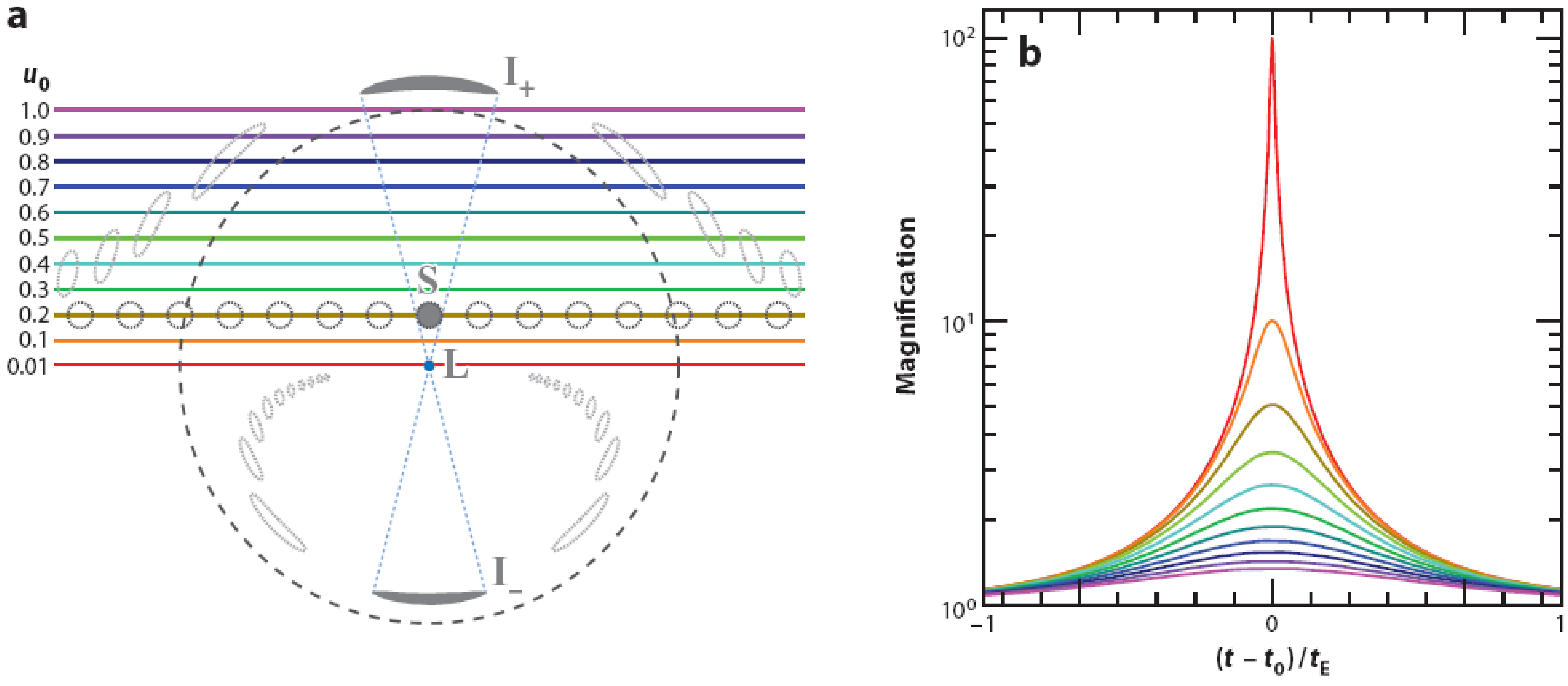
$$
\begin{aligned}
A(u) &= \frac{1}{u}\frac{1+u^2/2}{\sqrt{1+u^2/4}} \to \frac{1}{u}\left(1+\frac{3}{8}u^2\right) \qquad (u \ll 1) \\
A(u) &= \left(1-\frac{4}{(u^2+2)^2}\right)^{-1/2} \to 1+\frac{2}{(u^2+2)^2} \qquad (u \gg 1) \\
A(1) &= \frac{3}{\sqrt{5}} \simeq 1.34 \\
u(A) &= \sqrt{2[(1-A^{-2})^{-1/2}-1]}
\end{aligned}
$$

Simple Point Lens

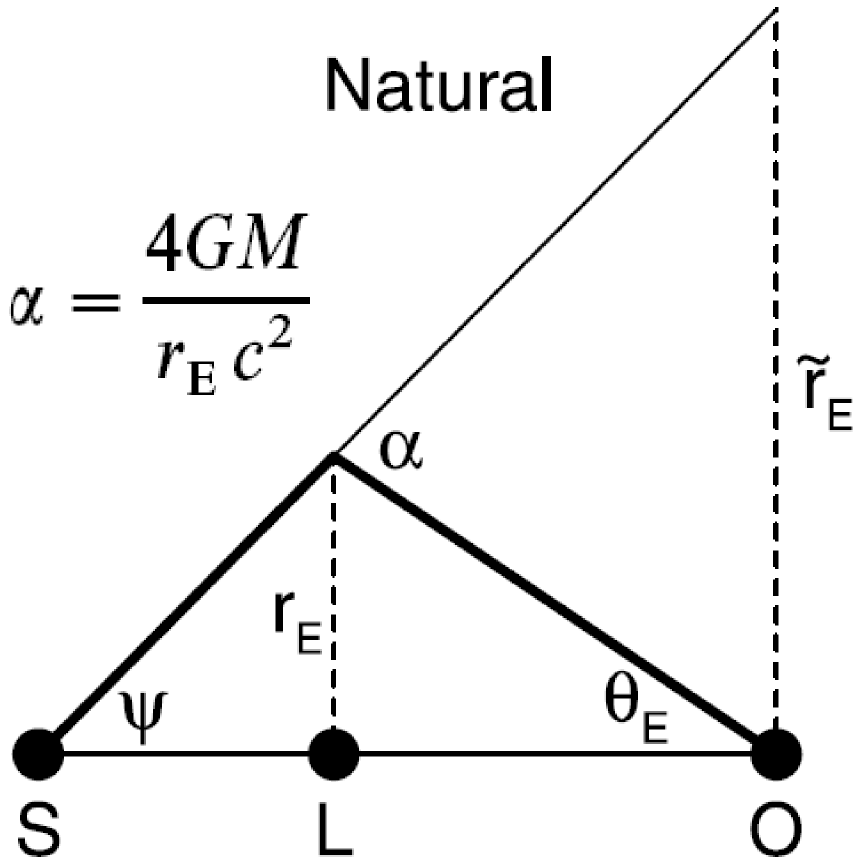
3 Features & 3 Parameters

3 Features	3 Parameters
Time of Peak	t_0
Height of Peak	u_0
Width of Peak	t_E

Point-Lens Light Curves



Relation of Mass and Distance to Lensing Observables



$$\alpha/\tilde{r}_E = \theta_E/r_E$$

$$\theta_E \tilde{r}_E = \alpha r_E = \frac{4GM}{c^2}$$

$$\theta_E = \alpha - \psi = \frac{\tilde{r}_E}{D_l} - \frac{\tilde{r}_E}{D_s} = \frac{\tilde{r}_E}{D_{rel}}$$

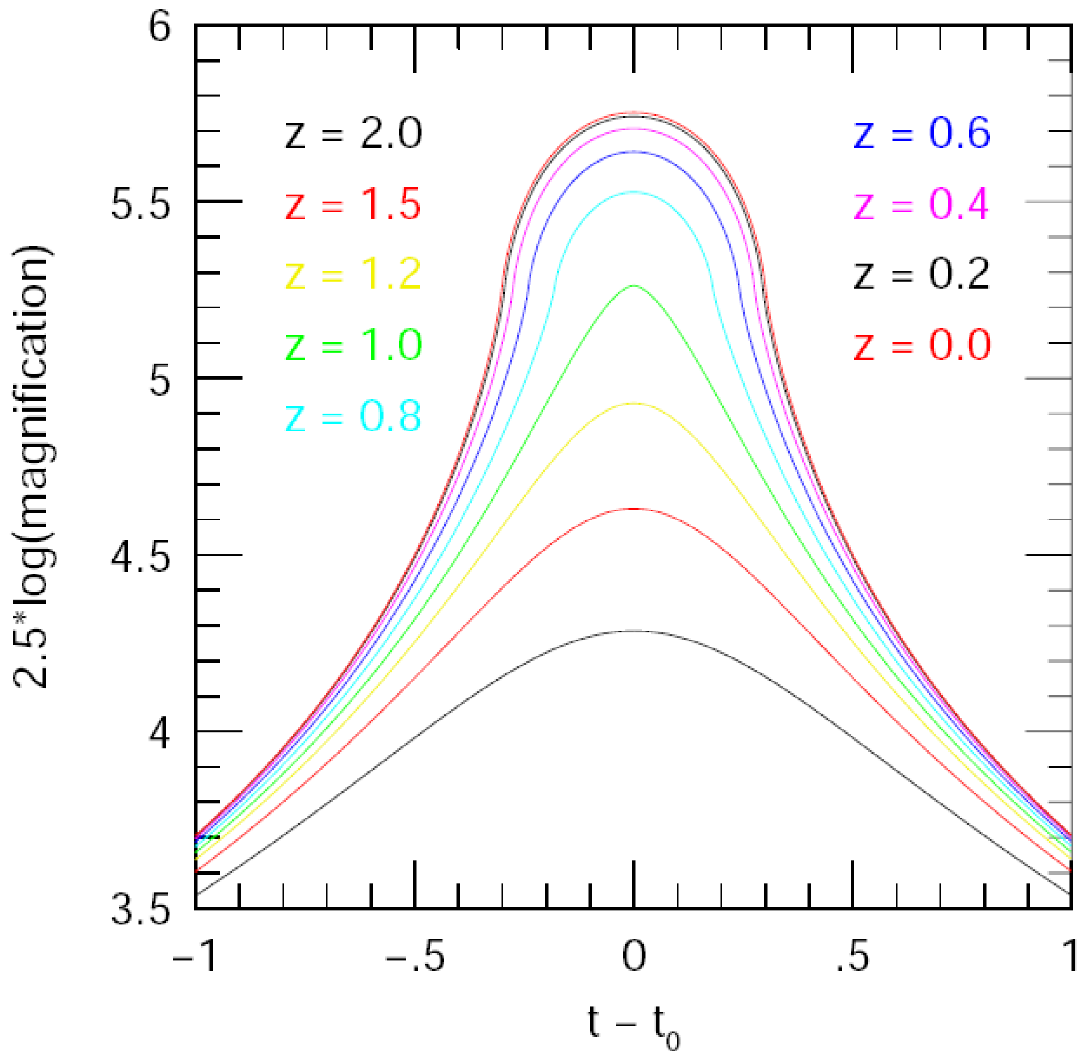
$$\tilde{r}_E = \sqrt{\frac{4GMD_{rel}}{c^2}} \qquad \theta_E = \sqrt{\frac{4GM}{D_{rel} c^2}}$$

Point Lens + Finite Source Effect

4 Features & 4 Parameters

4 Features	4 Parameters
Time of Peak	t_0
Height of Peak	u_0
Width of Peak	t_E
Width of Cap	$t_* = \rho * t_E$

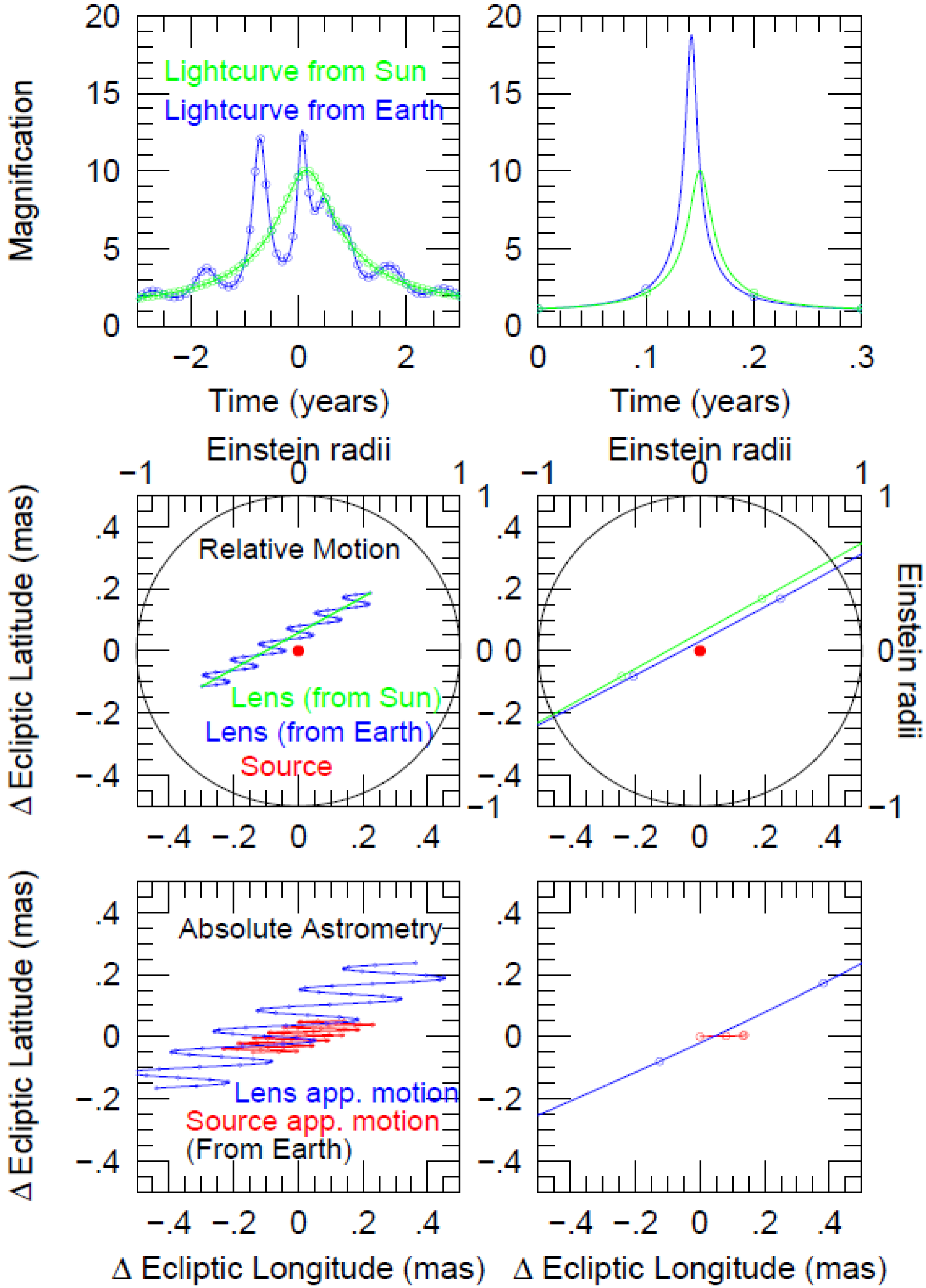
Finite Source "Attenuation"



Point Lens + Parallax

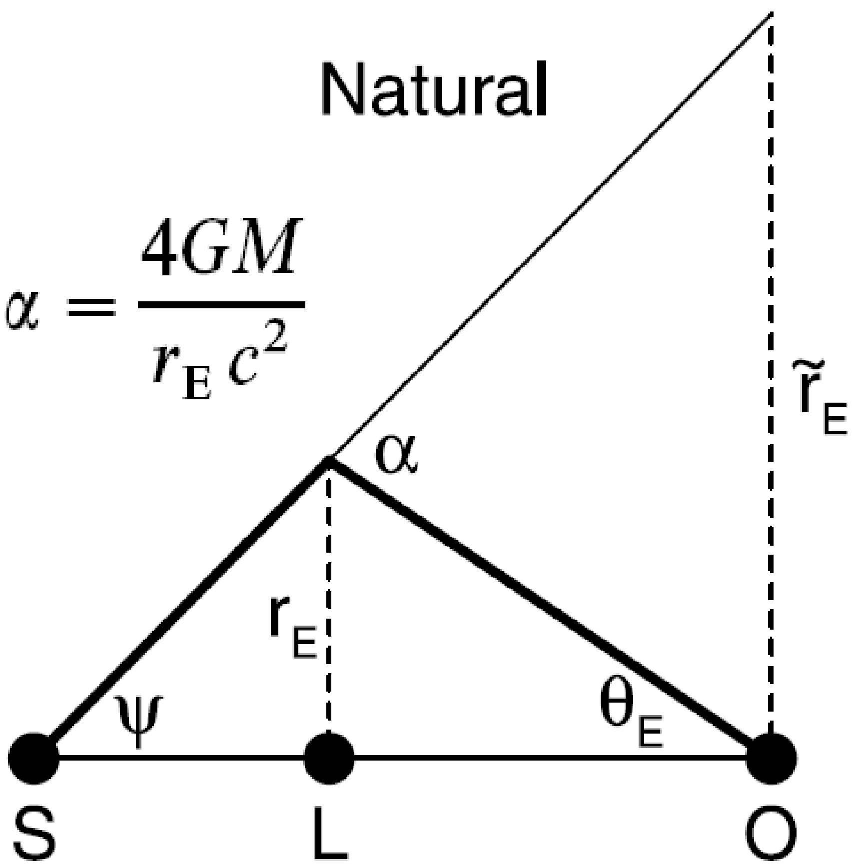
5 Features & 5 Parameters

Time of Peak	t_0
Height of Peak	u_0
Width of Peak	t_E
Symmetric Distortion	$\pi_{E,perp}$
Anti-Symmetric Distortion	$\pi_{E,parallel}$



Magnification
Lightcurve from Sun
Lightcurve from Earth
Time (years)
Time (years)
Einstein radii
Einstein radii
Relative Motion
Lens (from Sun)
Lens (from Earth)
Source
Δ Ecliptic Latitude (mas)
Einstein radii
Absolute Astrometry
Lens app. motion
Source app. motion
(From Earth)
Δ Ecliptic Latitude (mas)
Δ Ecliptic Longitude (mas)
Δ Ecliptic Longitude (mas)

Relation of Mass and Distance to Lensing Observables



$$\alpha = \frac{4GM}{r_E c^2}$$

$$\alpha/\tilde{r}_E = \theta_E/r_E$$

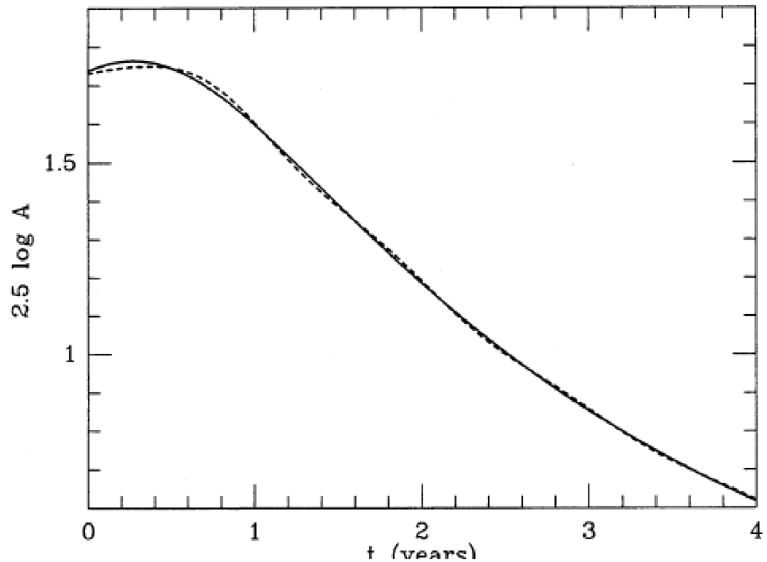
$$\theta_E \tilde{r}_E = \alpha r_E = \frac{4GM}{c^2}$$

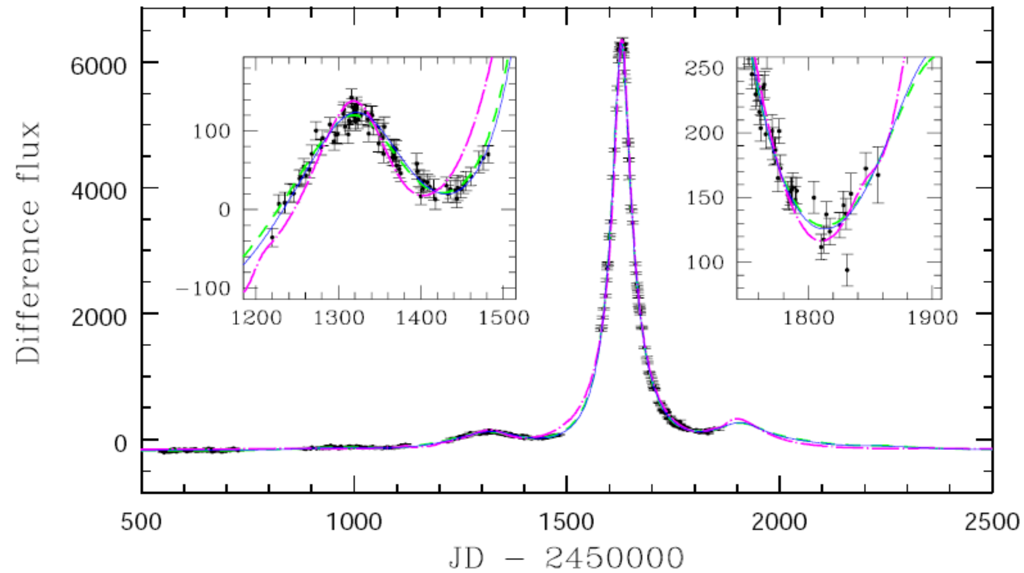
$$\theta_E = \alpha - \psi = \frac{\tilde{r}_E}{D_l} - \frac{\tilde{r}_E}{D_s} = \frac{\tilde{r}_E}{D_{rel}}$$

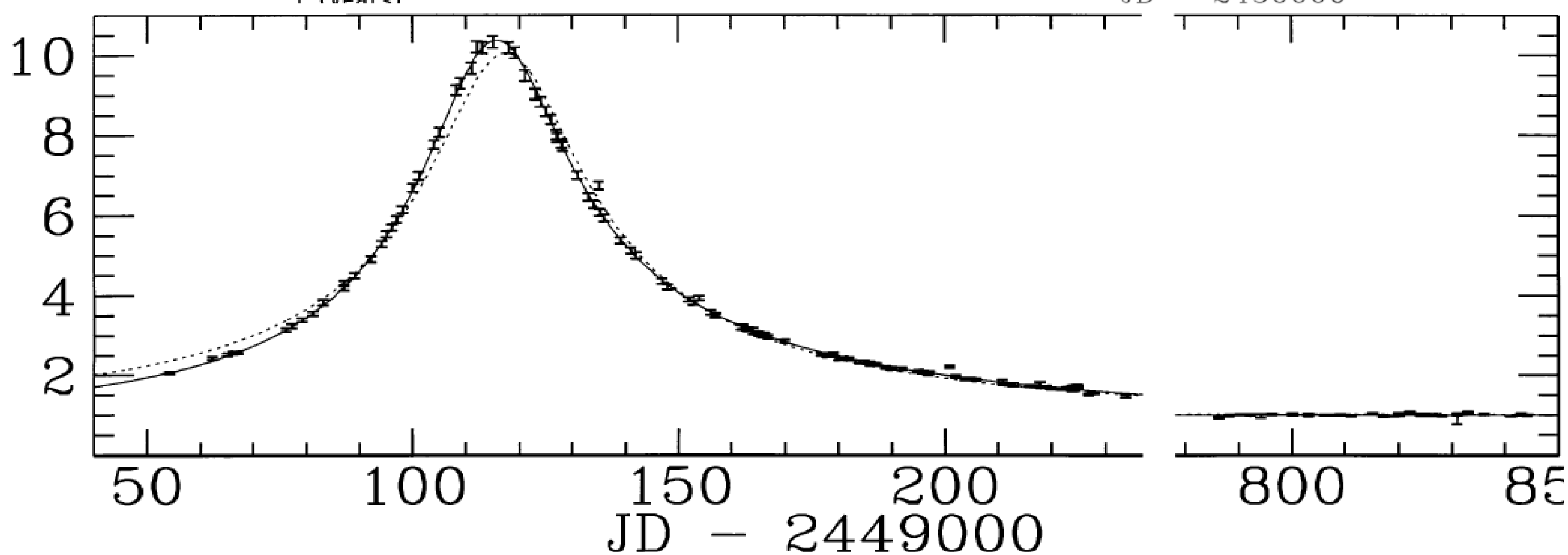
$$\tilde{r}_E = \sqrt{\frac{4GMD_{rel}}{c^2}} \qquad \theta_E = \sqrt{\frac{4GM}{D_{rel} c^2}}$$

Parallax Examples

Many Year, Few Year, <1 Year







Point Lens + Parallax + FS

6 Features & 6 Parameters

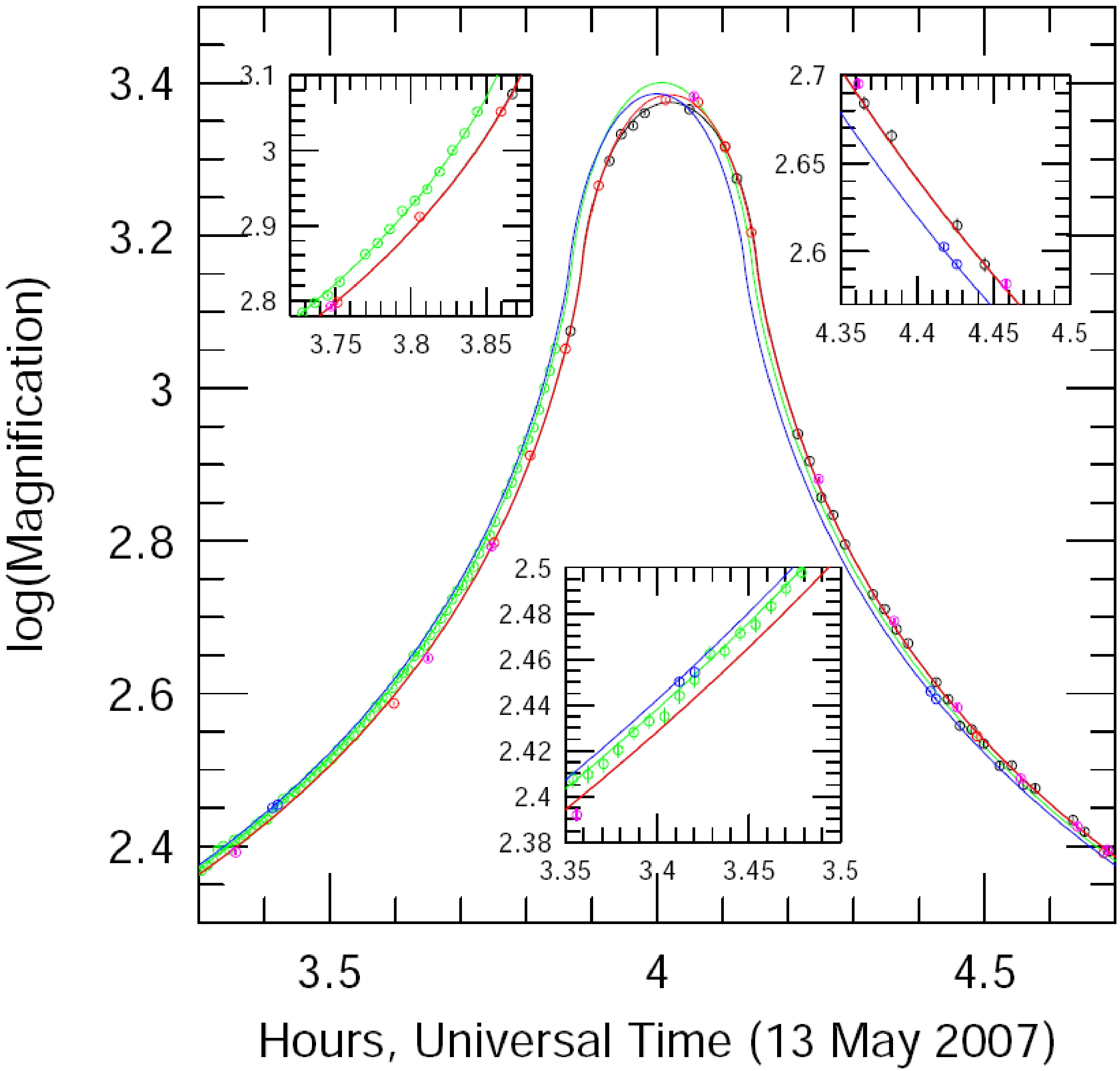
Time of Peak	t_0
Height of Peak	u_0
Width of Peak	t_E
Width of Cap	$t_* = \rho * t_E$
Symmetric Distortion	$\pi_{E,perp}$
Anti-Symmetric Distortion	$\pi_{E,parallel}$

Real Examples:

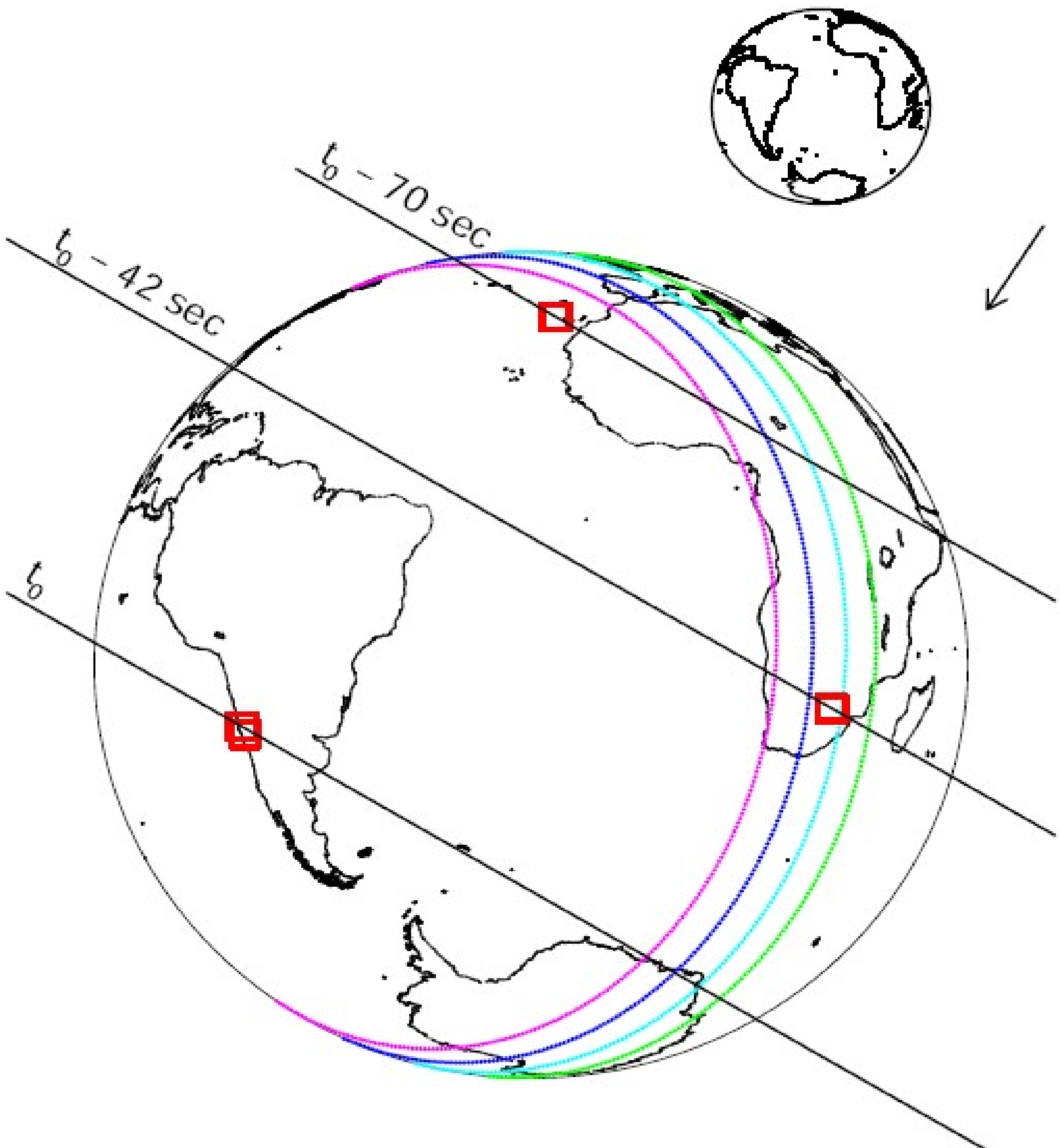
NONE

OGLE-2007-BLG-224

Canaries South Africa Chile



Terrestrial Parallax: Simultaneous Observations on Earth



Simple Planetary (G&L) Lenses

6 Features & 6 Parameters

6 Features	6 Parameters
Time of Peak	t_0
Height of Peak	u_0
Width of Peak	t_E
Time of Perturbation	Trajectory angle: α
Height of Perturbation	Planet-star separation: s
Width of Perturbation	Planet/star mass ratio: q

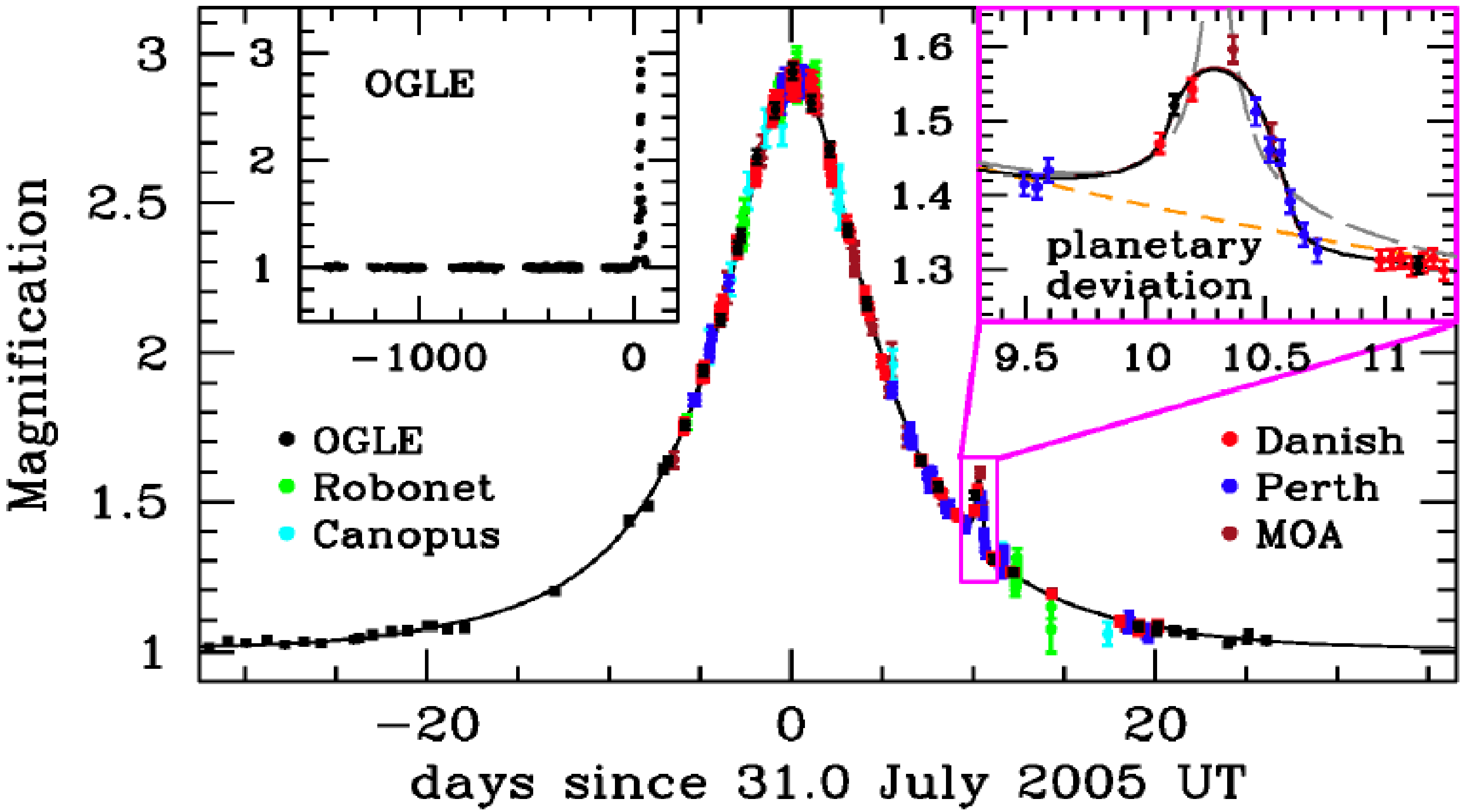
Planetary Lenses usually have FS

7 Features & 7 Parameters

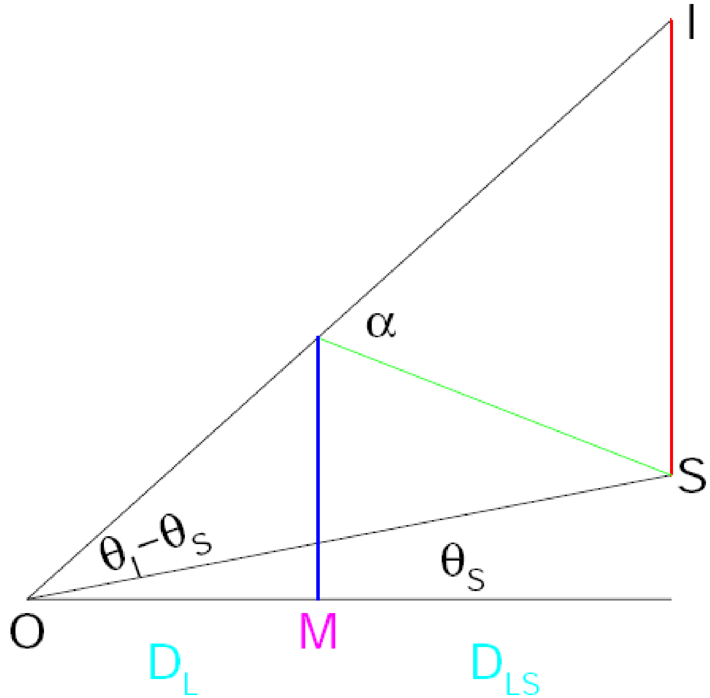
7 Features	7 Parameters
Time of Peak	t_0
Height of Peak	u_0
Width of Peak	t_E
Time of Perturbation	Trajectory angle: α
Height of Perturbation	Planet-star separation: s
Width of Perturbation	Planet/star mass ratio: q
Width of Caustic Cr.	t_* = ρ * t_E

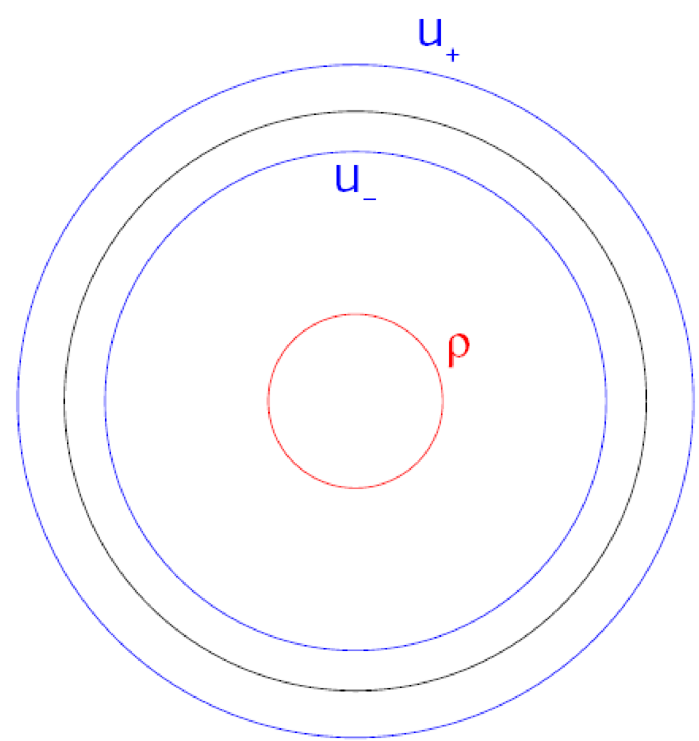
OGLE-2005-BLG-390

First Simple (G&L) Planetary Lens



Beaulieu et al. 2006, Nature, 439, 437





$$(\theta_I - \theta_S) D_S = \alpha D_{LS}$$

$$\alpha = 4GM/(D_L \theta_I c^2)$$

$$(\theta_I - \theta_S)\theta_I = \theta_E^{\,2} = (4GM/c^2)(D_{LS}/D_L D_S)$$

$$\theta_I/\theta_E = [u +/- (u^2 + 4)^{1/2}]/2; \quad u=\theta_S/\theta_E$$

Source Centered on Point Lens

$$A = \frac{\pi(u_+^2 - u_-^2)}{\pi\rho^2}, \qquad u_\pm = \frac{\rho \pm \sqrt{\rho^2 + 4}}{2}$$

$$A = \sqrt{1 + \frac{4}{\rho^2}} \rightarrow 1 + \frac{2}{\rho^2}, \qquad \rho \equiv \frac{\theta_*}{\theta_{\rm E}}$$

Conjecture for Big Source on Planet Caustic

$$A_p = 2\left(\frac{\theta_{{\rm E},p}}{\theta_*}\right)^2$$

Plus Simple Timing Argument

$$\frac{t_p}{t_{\rm E}} = \frac{\theta_*}{\theta_{\rm E}}$$

Yields Mass-Ratio Estimate

$$q = \frac{M_p}{M} = \frac{\theta_{{\rm E},p}^2}{\theta_{\rm E}^2} = \frac{\theta_{{\rm E},p}^2}{\theta_*^2}\frac{\theta_*^2}{\theta_{\rm E}^2} = \frac{A_p}{2}\frac{t_p^2}{t_{\rm E}^2}$$

Mass-Ratio Estimate
a la Gould & Loeb

q=(A_p/2)(t_p/t_E)^2

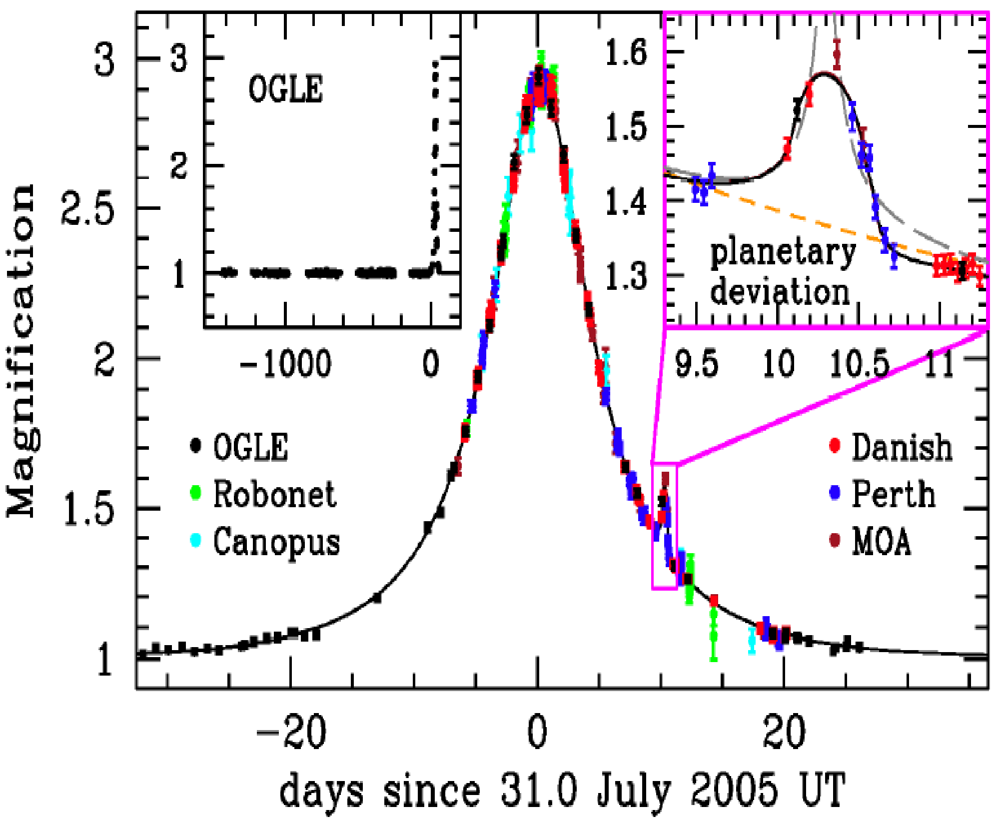
A_p = 0.2

t_p =0.3 day

t_E=10 day

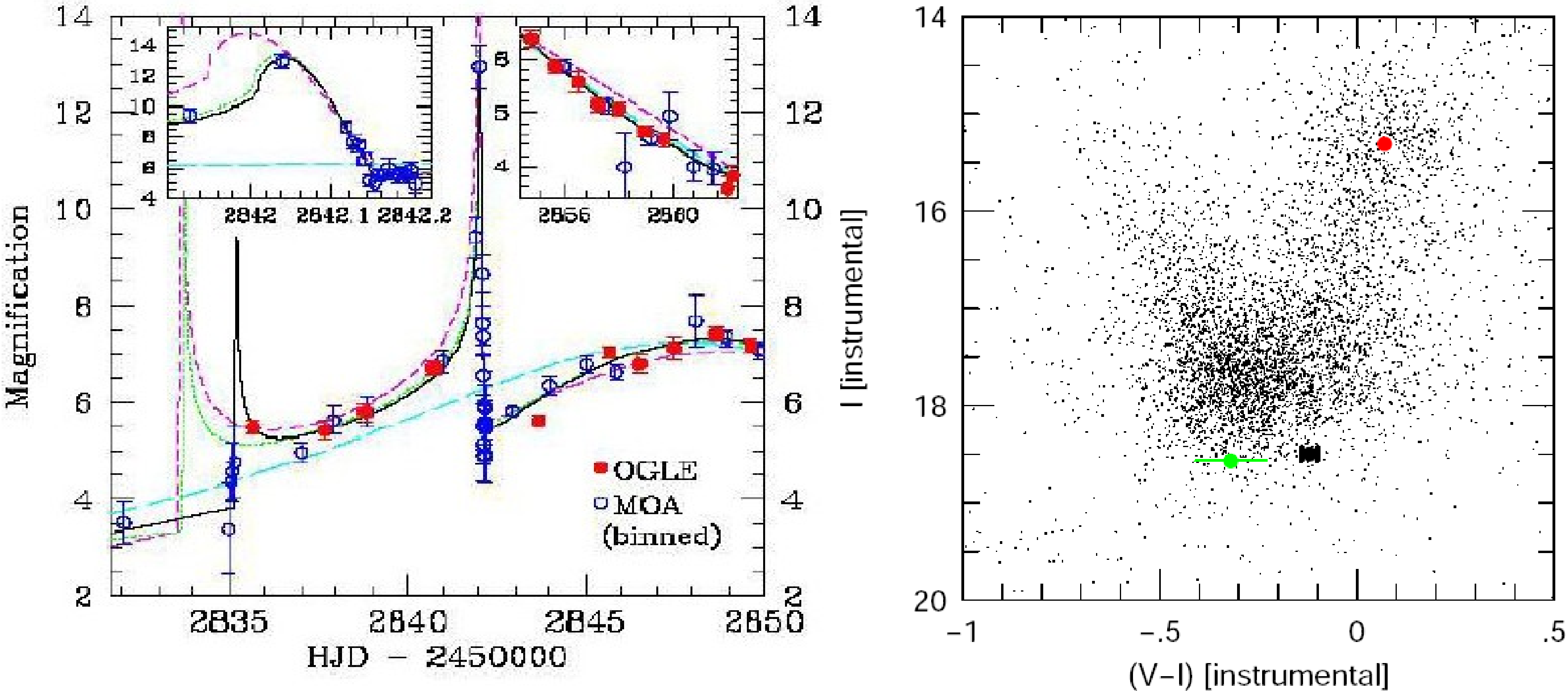
q=9e-5

q_actual = 8e-5

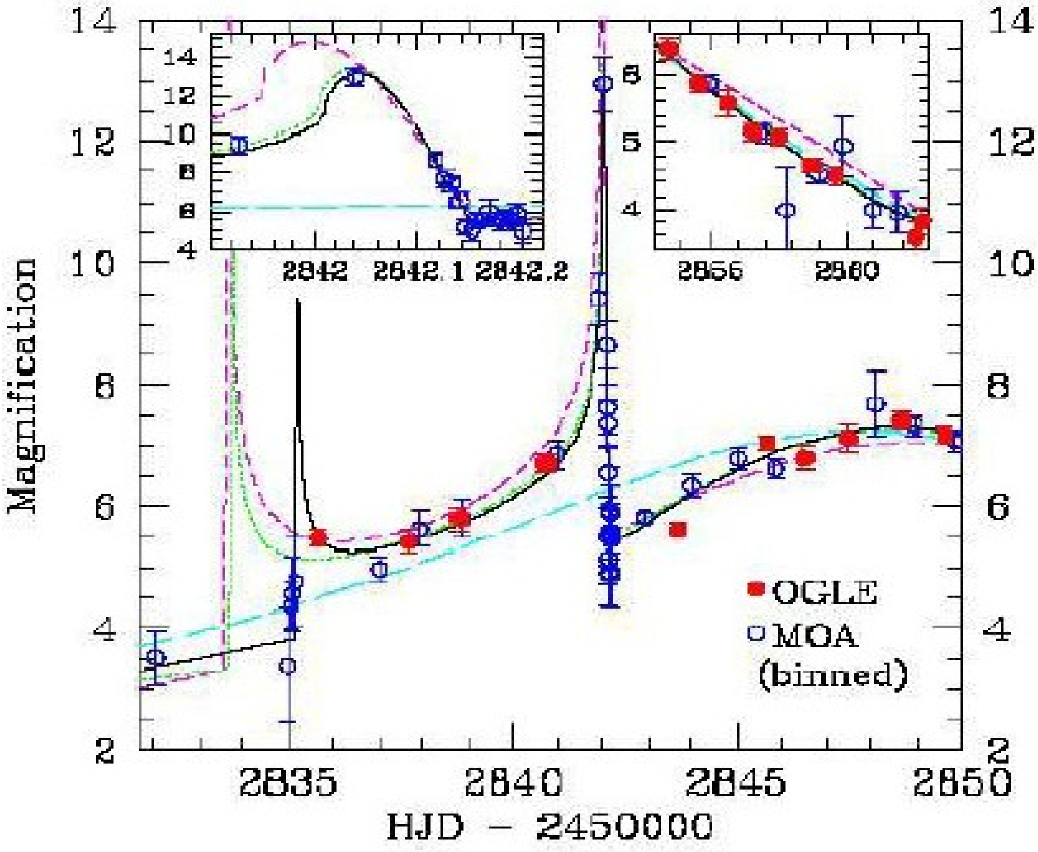


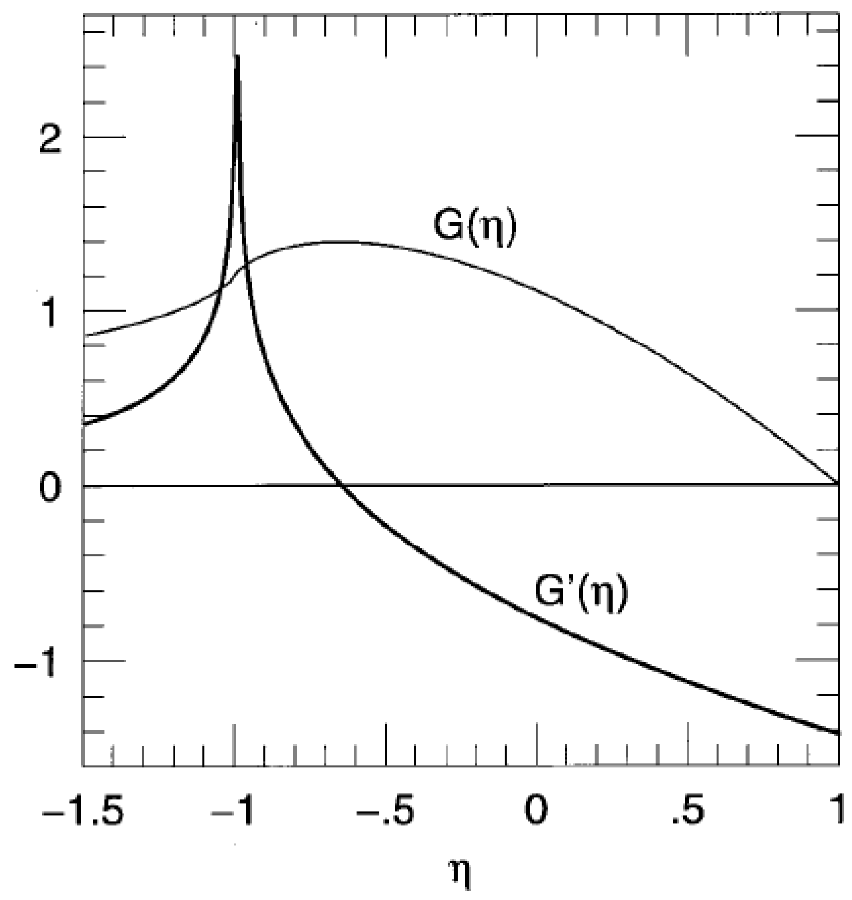
First Microlensing Planet

Pronounced Finite Source Effects

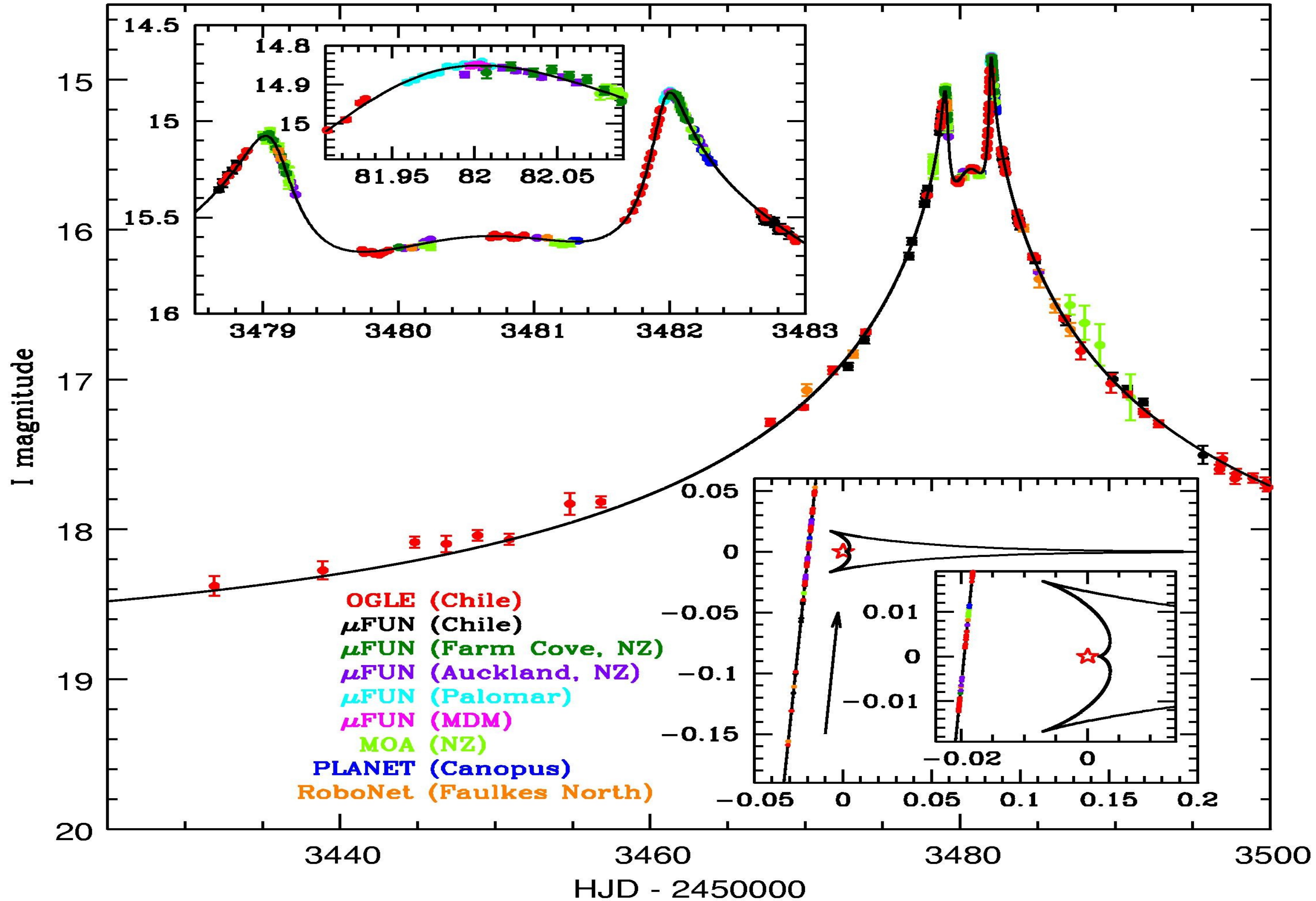


First Microlensing Planet
Perfect Fold Caustic Crossing





Second Microlensing Planet
Weak Finite Source Effects



Udalski et al. 2005, ApJ, 628, L109

Planet Lenses Often Have Parallax

9 Features & 9 Parameters

9 Features	9 Parameters
3 Point-Lens	t_0, u_0, t_E
Time of Perturbation	Trajectory angle: α
Height of Perturbation	Planet-star separation: s
Width of Perturbation	Planet/star mass ratio: q
Width of Caustic Cr.	$t_* = \rho * t_E$
Symmetric Distortion	$\pi_{E,perp}$
Anti-symmetric Dist.	$\pi_{E,parallel}$

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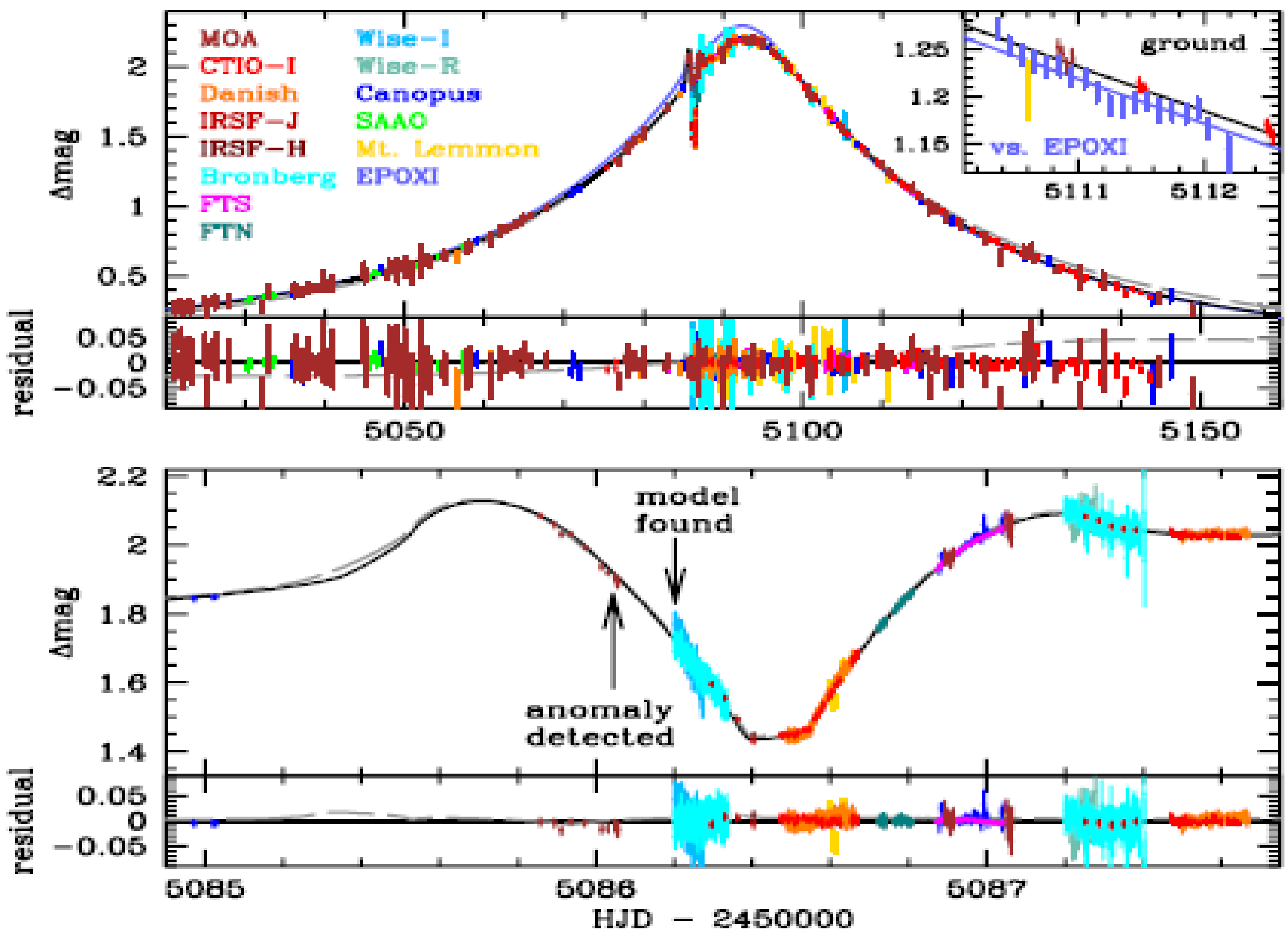
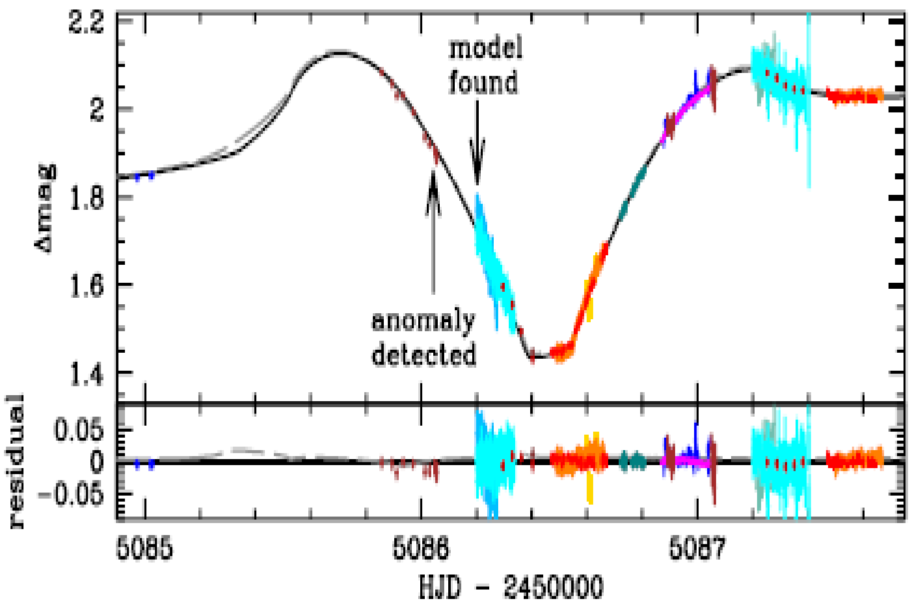
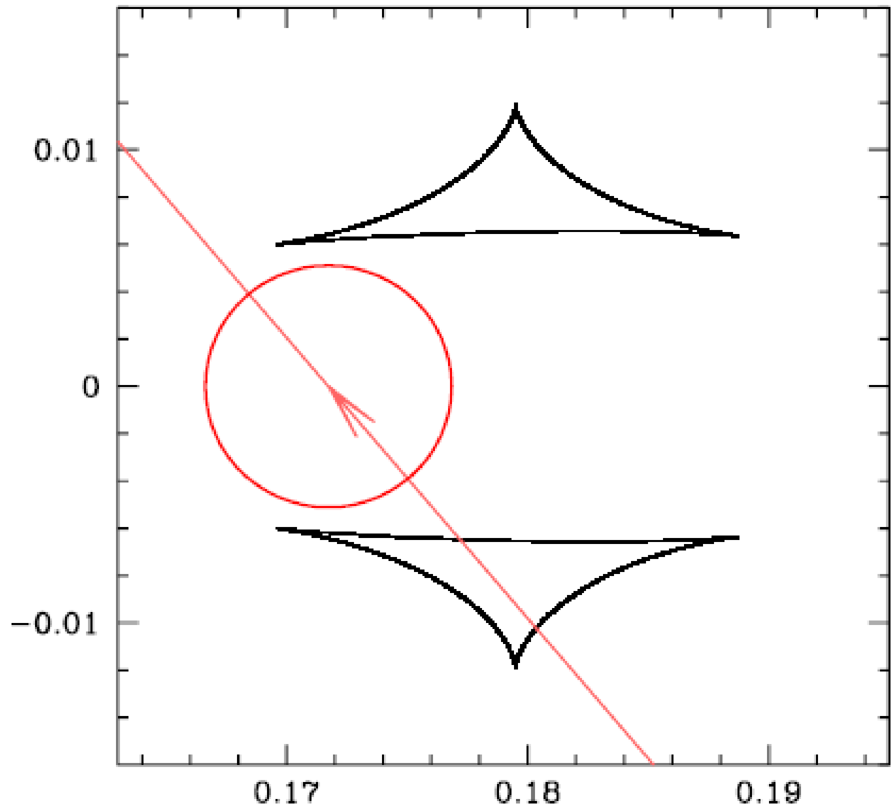
Parallax + Finite Source



Muraki et al. 2011, ApJ, 741, 22

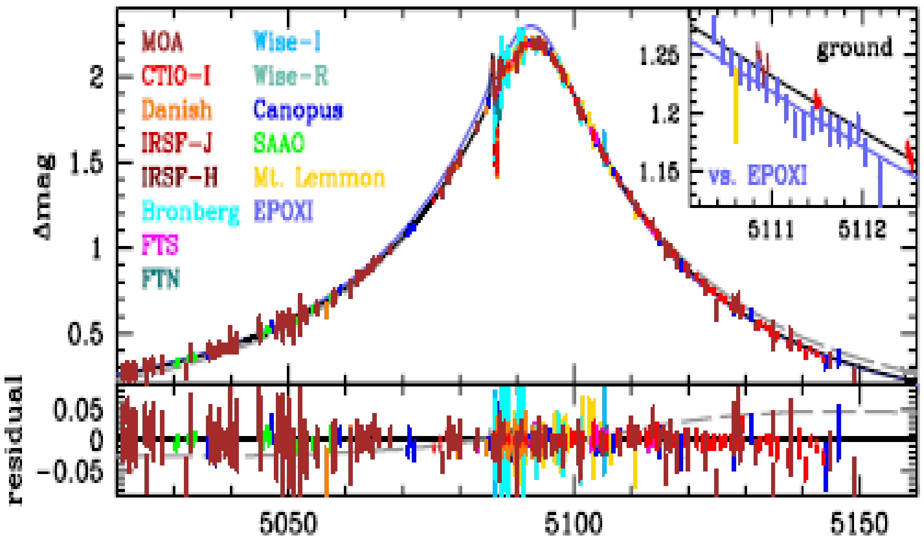
ρ Well-Measured from “Dip”

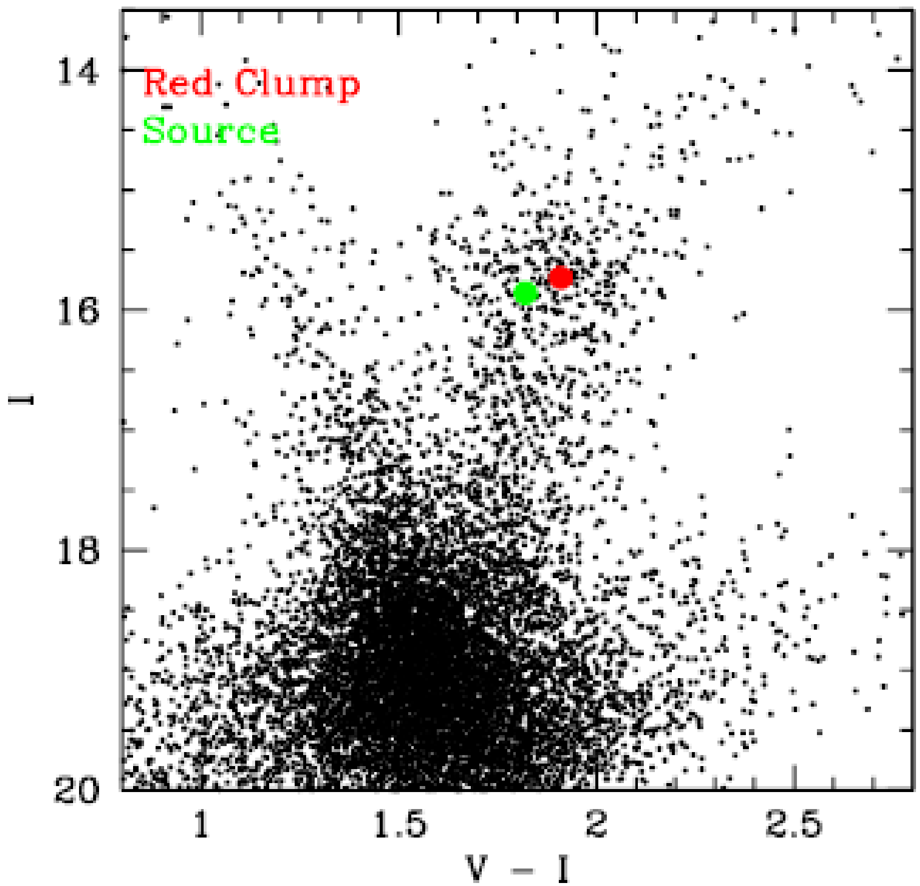




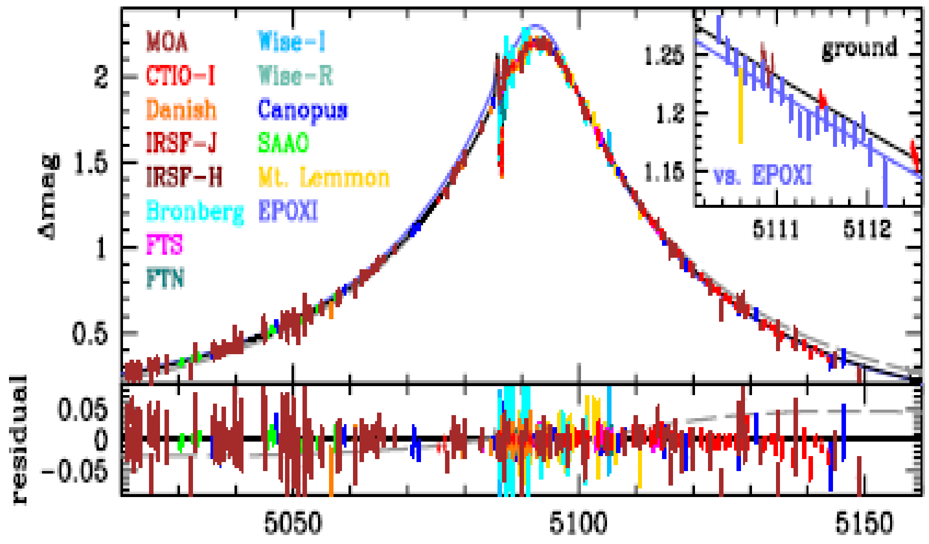
θ_* Well-measured from lightcurve

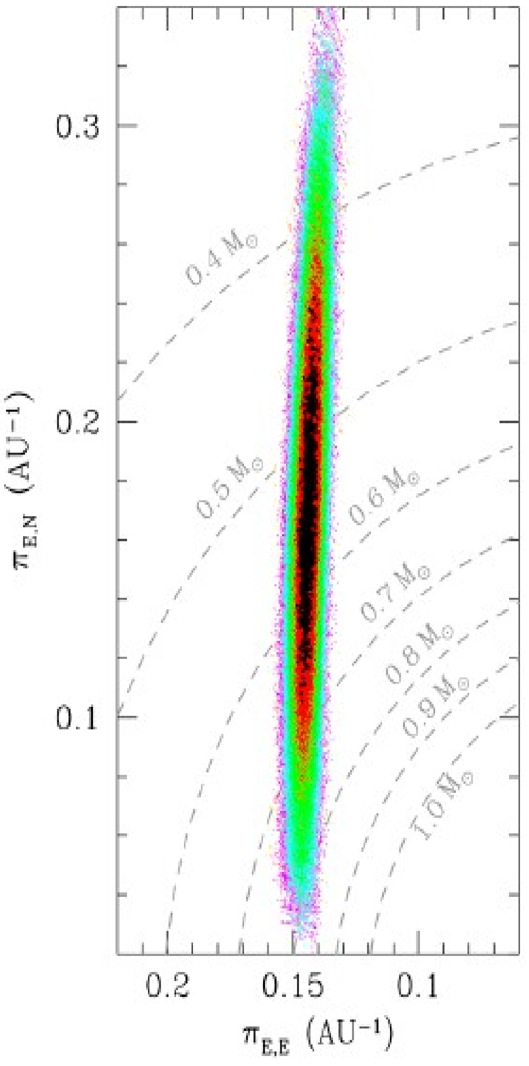
$$\Longrightarrow \theta_E = \theta_*/\rho$$





π_E semi-measured from lightcurve





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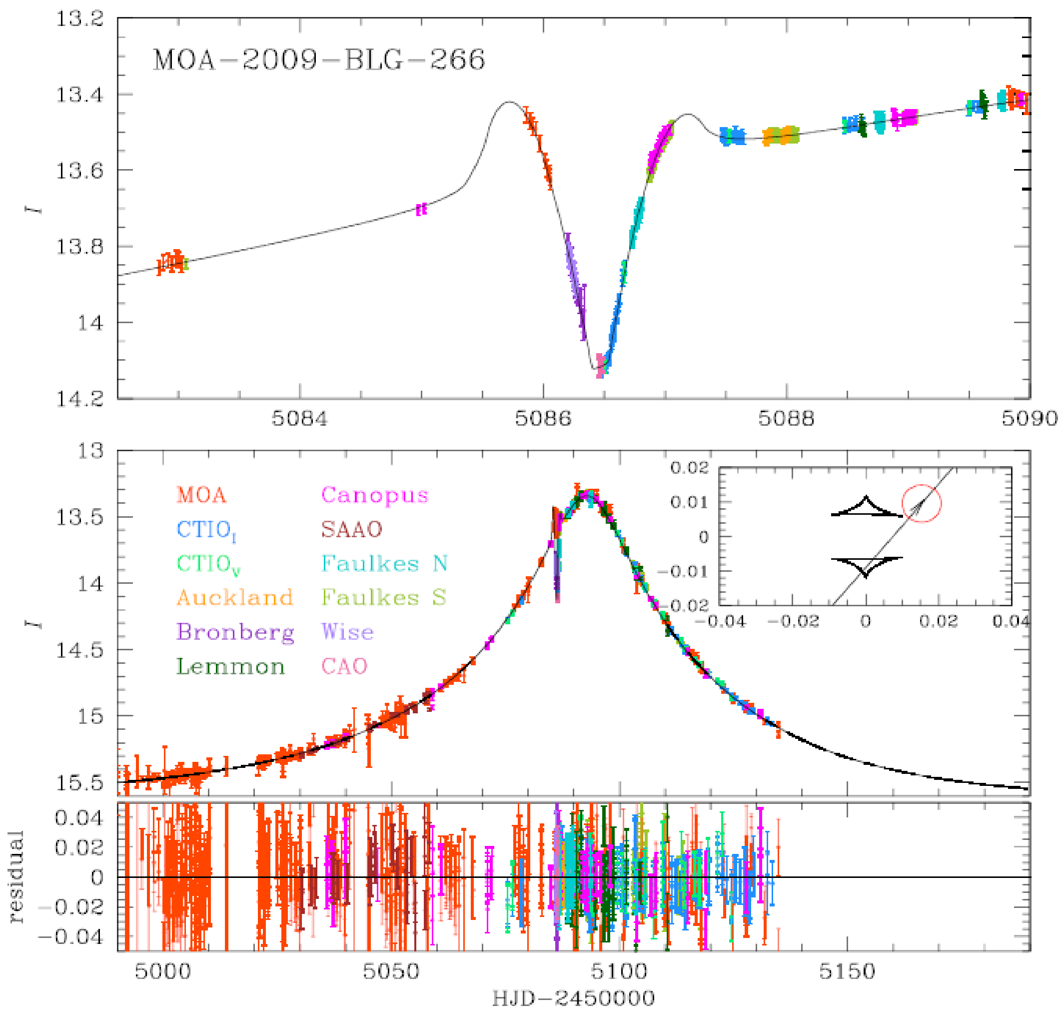
Minor Image Planetary Caustic



Preliminary Model (Cheongho Han)

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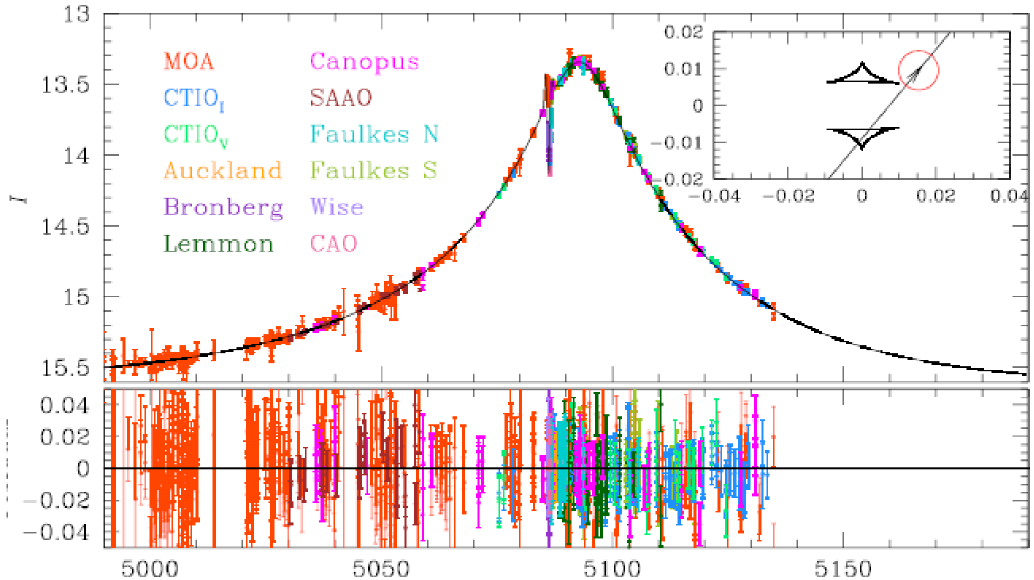

Main Parameters

$$t_0 = 5093.1 \qquad \text{[inspection]}$$

$$u_0 \simeq A_{\rm max}^{-1} = 10^{0.4(I_{\rm peak} - I_{\rm base})} = 0.132$$

$$[I_{\rm peak} = 13.35, \qquad I_{\rm base} = 15.55]$$

$$t_{{\rm E},1} = \frac{t_{\rm eff}}{u_0} = \frac{t_0 - t_{1/2,-}}{u_0} = \frac{5093.1 - 5084.7}{0.132} = 64\,\text{day}$$

$$t_{1/2,-} = t[I = I_{\rm peak} + 2.5\log 2^{1/2}] = t[I = 13.73] = 5084.7$$

$$t_{{\rm E},2} = t_0 - t_{{\rm ring},-} = 5093 - 5036 = 57\,\text{day}$$

$$t_{{\rm ring},-} = t[I = I_{\rm base} - 2.5\log(9/5)^{1/2}] = t[I = 15.23] = 5036$$

$$t_{\rm E} \to \frac{t_{{\rm E},1} + t_{{\rm E},2}}{2} = 60\,\text{day}$$

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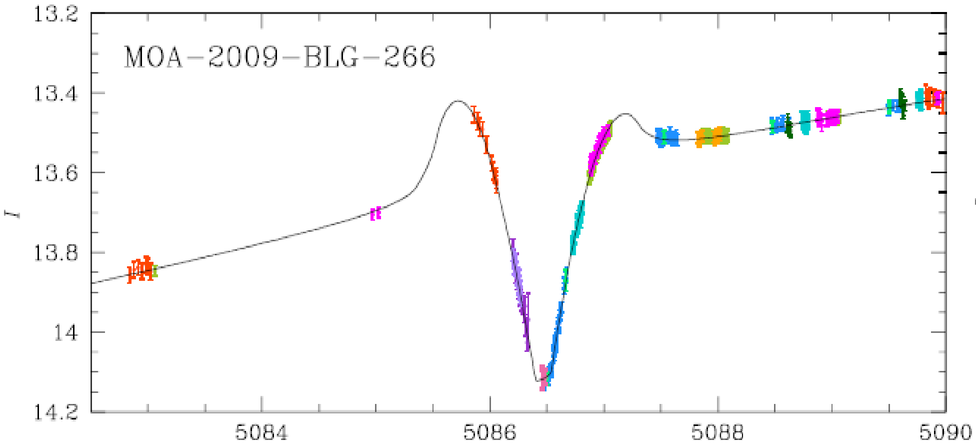
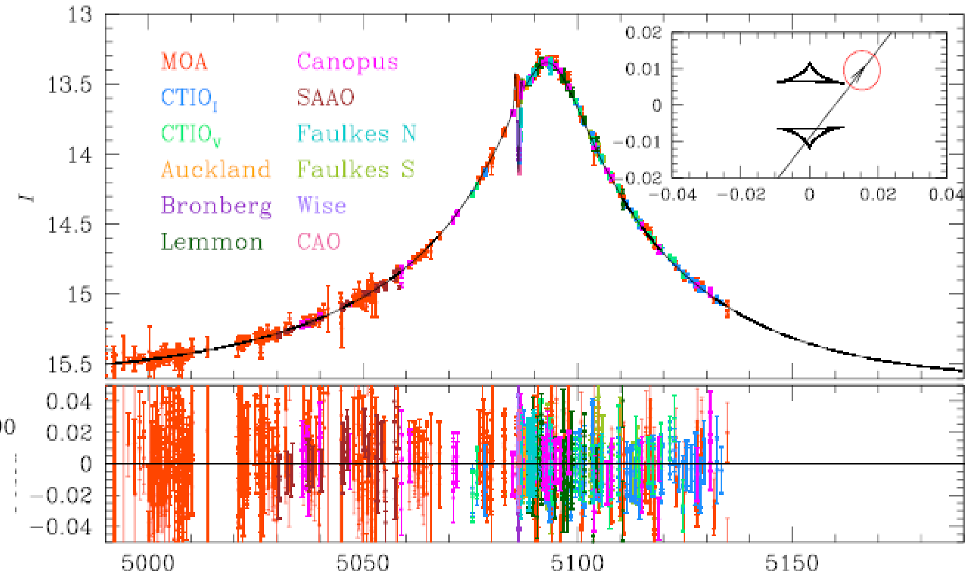




$$t_{0,\text{planet}} = 5086.5 \qquad \tau_{\text{planet}} = \frac{t_0 - t_{0,\text{planet}}}{t_{\text{E}}} = \frac{6.6}{60} = 0.11$$

$$u_{\text{planet},1} = A^{-1}_{\text{planet}} = 10^{0.4(I_{\text{plan}} - I_{\text{base}})} = 0.154 \qquad [I_{\text{planet}} = 13.58]$$

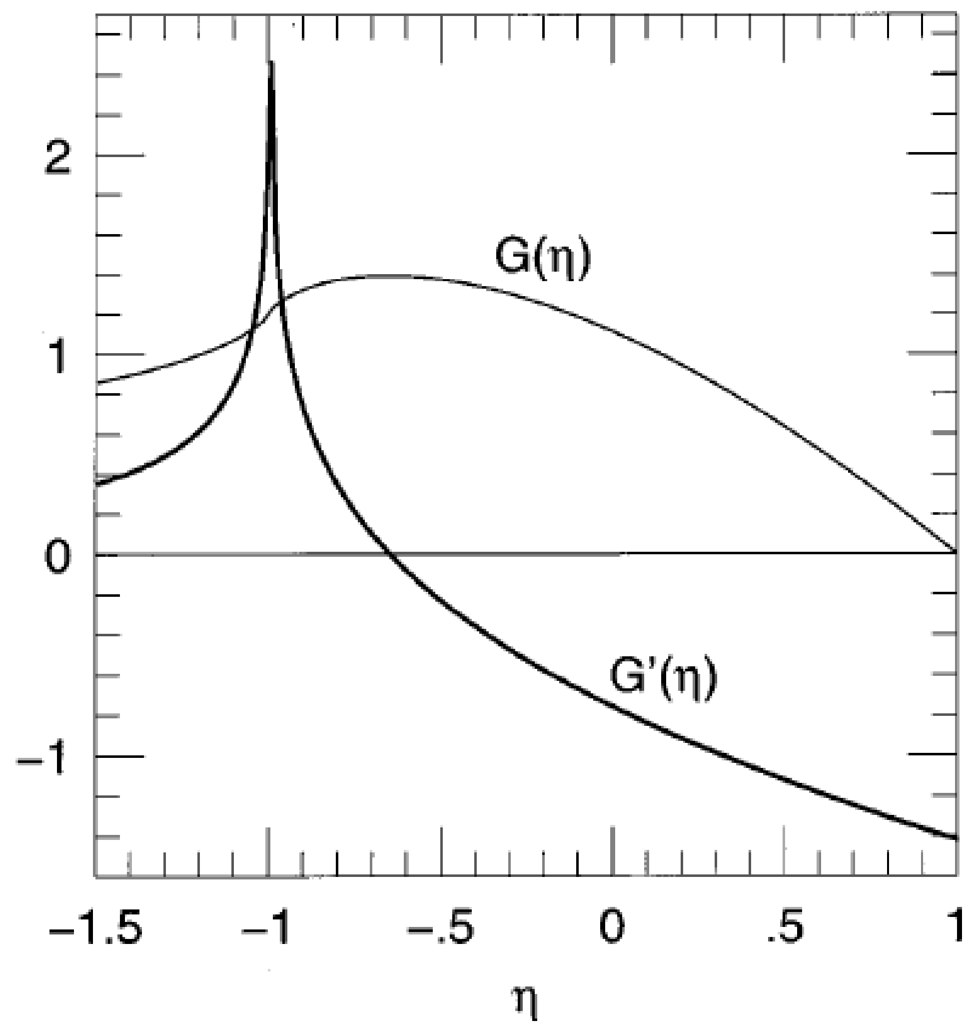
$$u_{\text{planet},2} = \sqrt{u_0^2 + \tau^2_{\text{planet}}} = 0.172$$

$$u_{\text{planet}} = \frac{u_{\text{planet},1} + u_{\text{planet},2}}{2} = 0.163$$

$$s = \frac{-u_{\text{planet}} + \sqrt{u^2_{\text{planet}} + 4}}{2} = 0.922$$

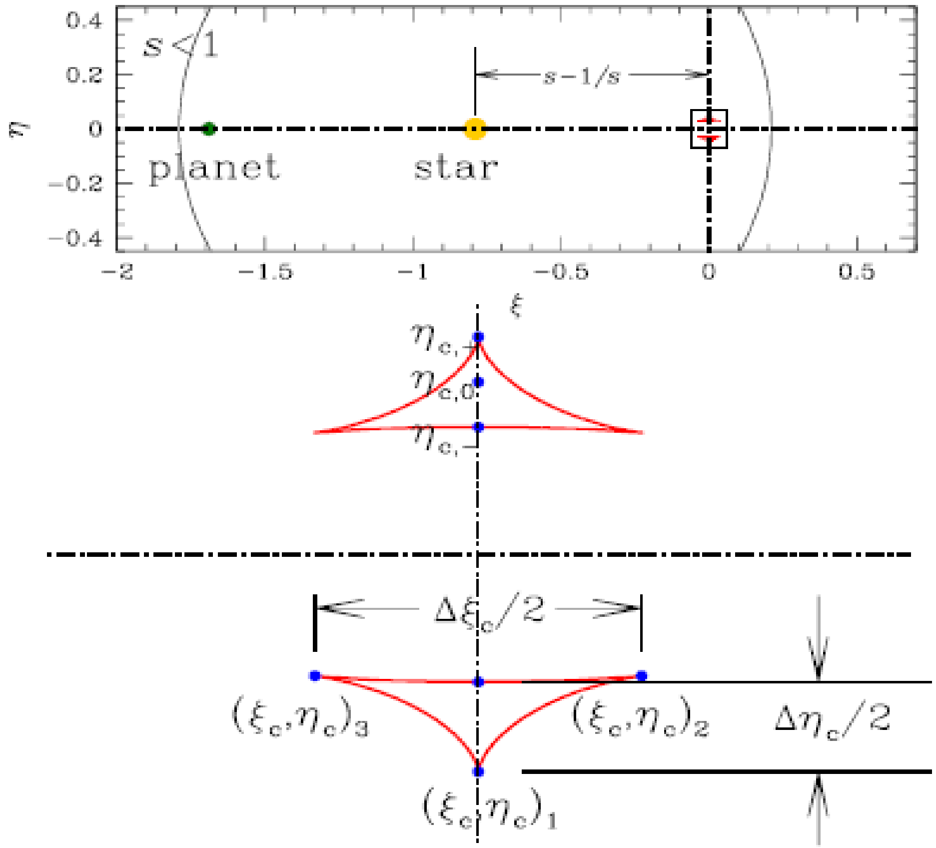
$$\alpha = \sin^{-1} \frac{u_0}{u_{\text{planet}}} = 54^\circ$$

Generic Caustic Exit



Gould & Andronov 1999, ApJ, 516, 236

Minor Image Analytic Formulae



$$\eta_{c,-} = 2q^{1/2}(1-s^2)^{1/2}/s$$

Han 2006, ApJ, 638, 1080

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Planet Parameters II: harder

$$t_{\mathrm{cross},1} = \frac{t_{\mathrm{planet-peak},1} - t_{\mathrm{planet-trough},1}}{1.7} = 0.41\,\mathrm{day}$$

$$t_{\mathrm{cc},1} = t_{\mathrm{planet-peak},1} + 0.7 * t_{\mathrm{cross},1} = 5085.98$$

$$t_{\mathrm{planet-peak},1} = 5085.7, \qquad t_{\mathrm{planet-trough},1} = 5086.4$$

$$t_{\mathrm{cross},2} = \frac{t_{\mathrm{planet-peak},2} - t_{\mathrm{planet-trough},2}}{-1.7} = 0.38\,\mathrm{day}$$

$$t_{\mathrm{cc},2} = t_{\mathrm{planet-peak},2} - 0.7 * t_{\mathrm{cross},2} = 5086.93$$

$$t_{\mathrm{planet-peak},2} = 5087.2, \qquad t_{\mathrm{planet-trough},1} = 5086.55$$

$$t_{\mathrm{cross}} = \frac{t_{\mathrm{cross},1} + t_{\mathrm{cross},2}}{2} = 0.397\,\mathrm{day}$$

$$\Delta u = \frac{t_{\mathrm{cc},2} - t_{\mathrm{cc},1}}{t_{\mathrm{E}}} \sin\alpha = 0.0128$$

$$\Delta u = 4\sqrt{\frac{q u_{\mathrm{planet}}}{s}} \Rightarrow q = \frac{s}{u_{\mathrm{planet}}}\left(\frac{\Delta u}{4}\right)^2 = 5.8 \times 10^{-5}$$

$$t_* = t_{\mathrm{cross}} \sin\alpha = 0.32\,\mathrm{day}, \qquad \rho = \frac{t_*}{t_{\mathrm{E}}} = 5.3 \times 10^{-3}$$

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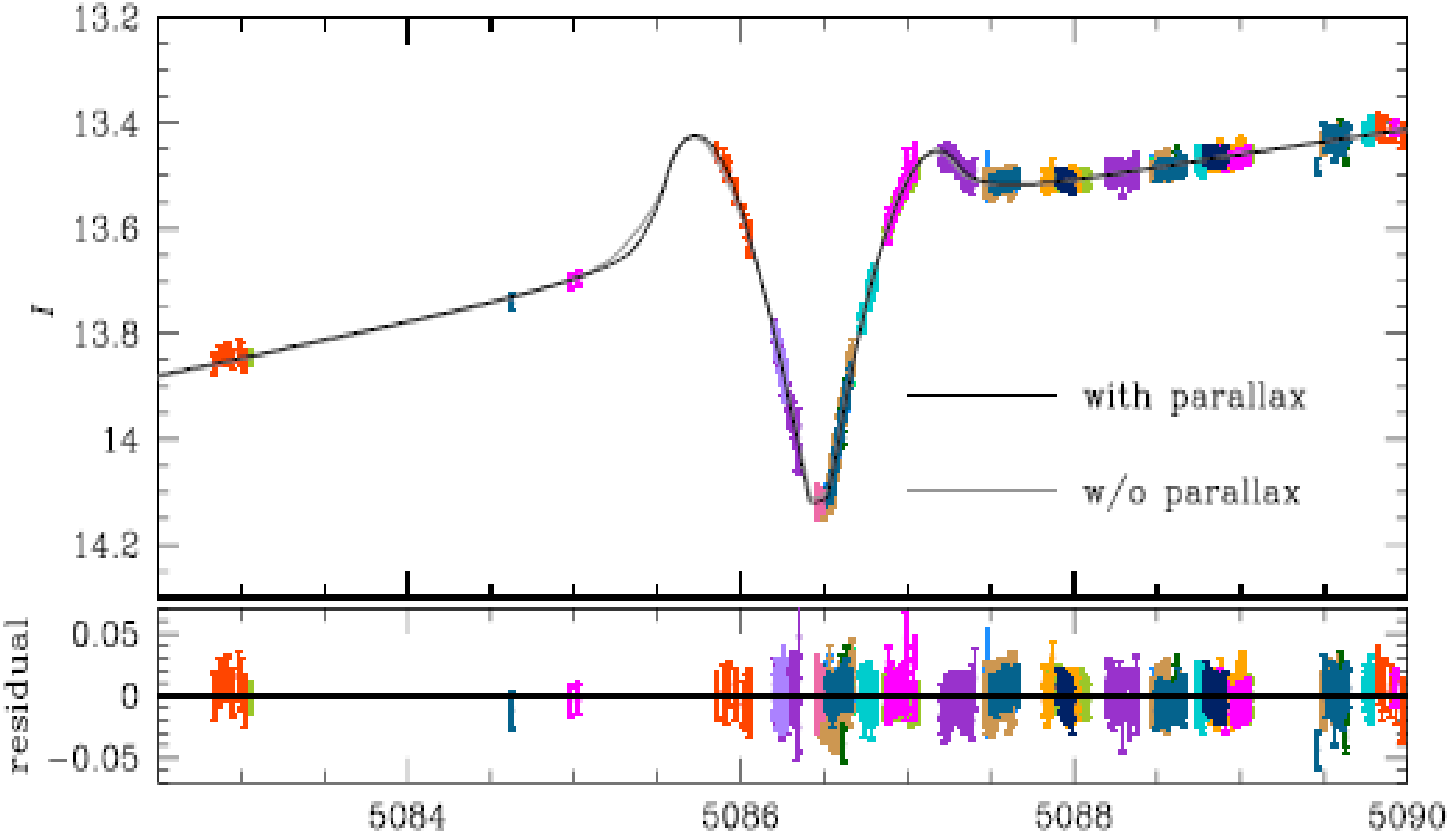

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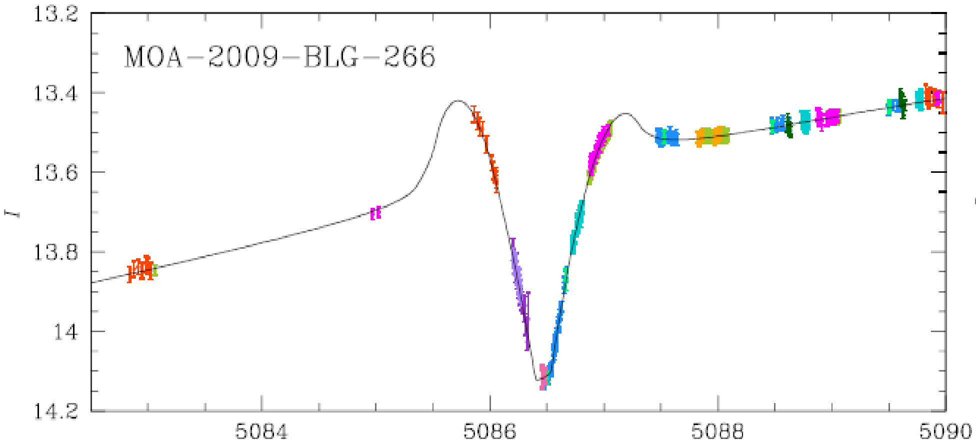


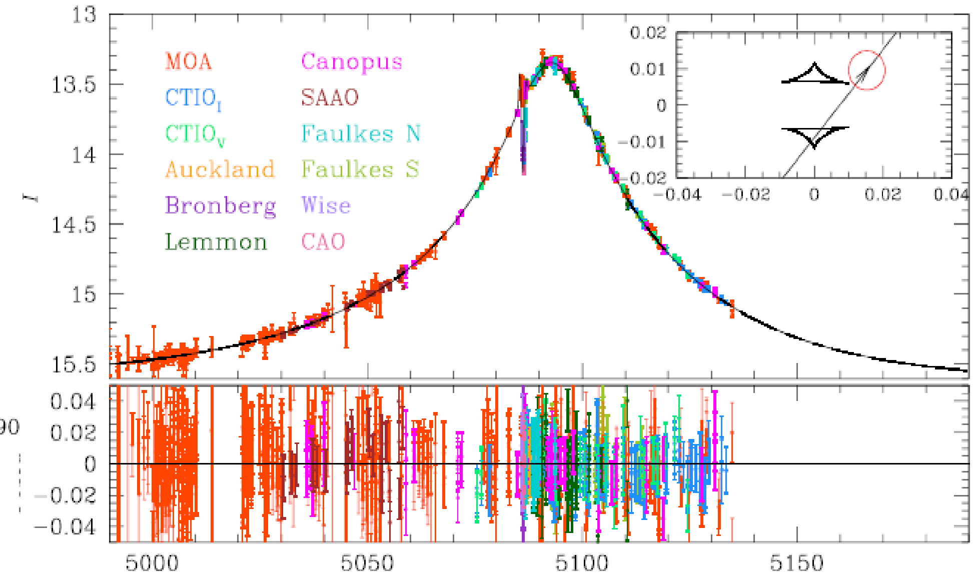


TABLE 1

MB09266: Eye vs. Computer

Parameter	Eye	Computer
t_0	5093.1	5093.07
u_0	0.13	0.13
t_E	60 d	60.2 d
q	5.8×10^{-5}	5.4×10^{-5}
s	0.922	0.914
α	54°	51°
ρ	5.3×10^{-3}	5.3×10^{-3}

Minor Image Test

$$\frac{A_{\rm trough}}{A_{\rm planet}} = 10^{0.4(I_{\rm planet} - I_{\rm trough})}$$

$$= 10^{0.4(13.58-14.02)} = 0.667$$

$$\frac{A_{\rm planet}+1}{2A_{\rm planet}} = 0.657$$

Planet Lenses: + Projected Motion

11 Features & 11 Parameters

3 Point-Lens

3 Binary-Lens

Width of Caustic Cr.

Symmetric Distortion

Anti-symmetric Dist.

Rotational Motion

Radial Motion

t_0, u_0, t_E

α_0, s_0, q

t_* = ρ * t_E

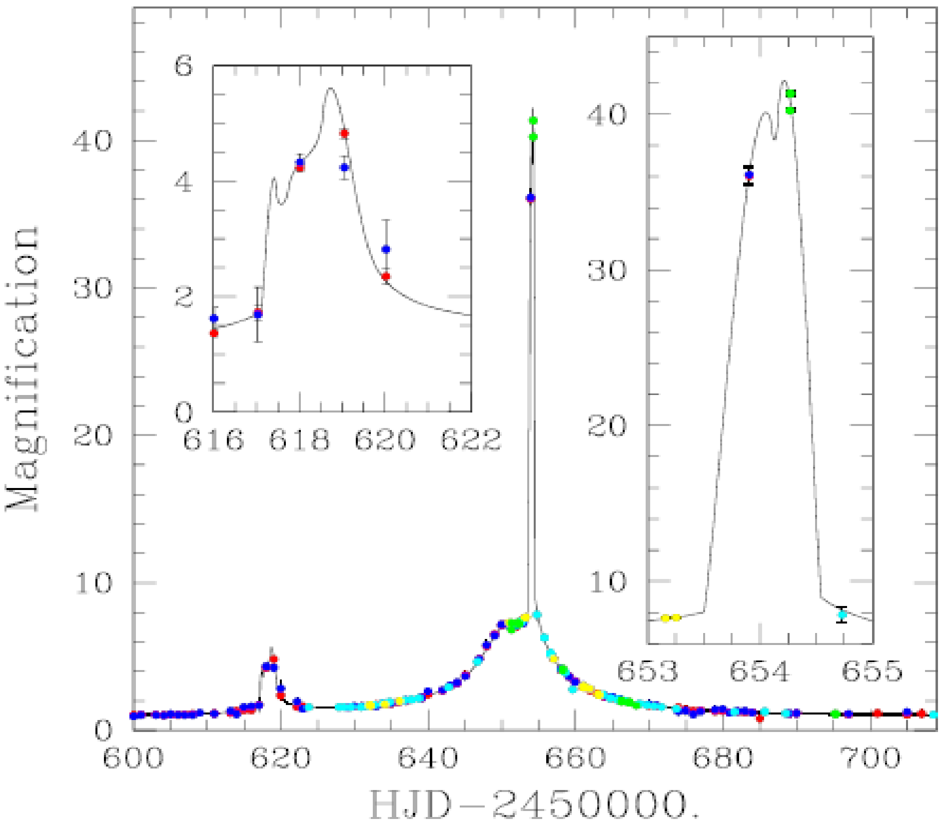
π_E,perp

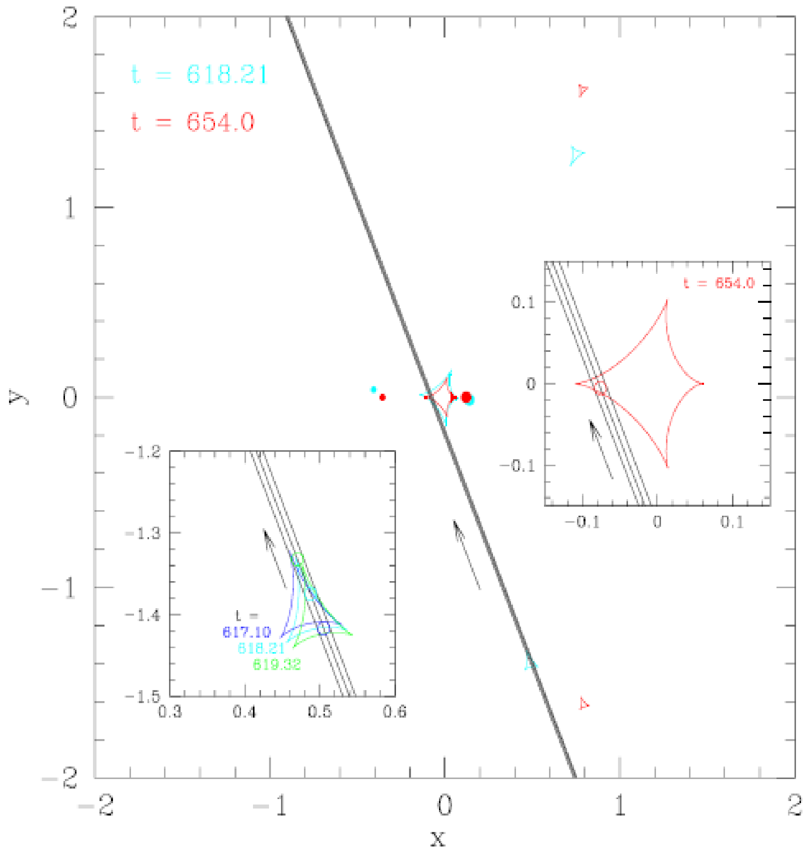
π_E,parallel

γ_perp = dα/dt

γ_parallel = (ds/dt)/s_0

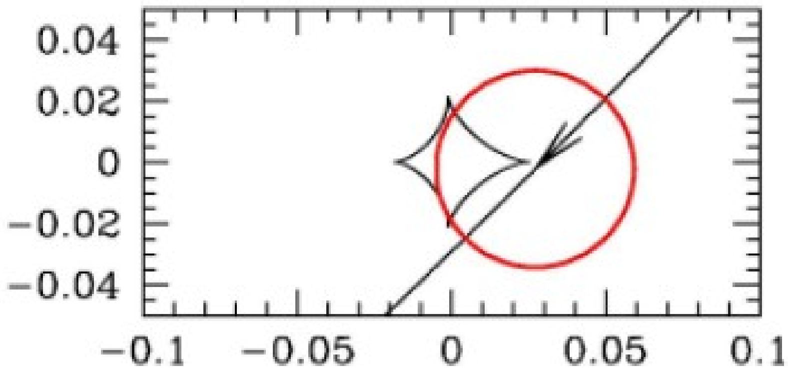
Macho 97-41: Obvious Orbital Motion
(But No Parallax)

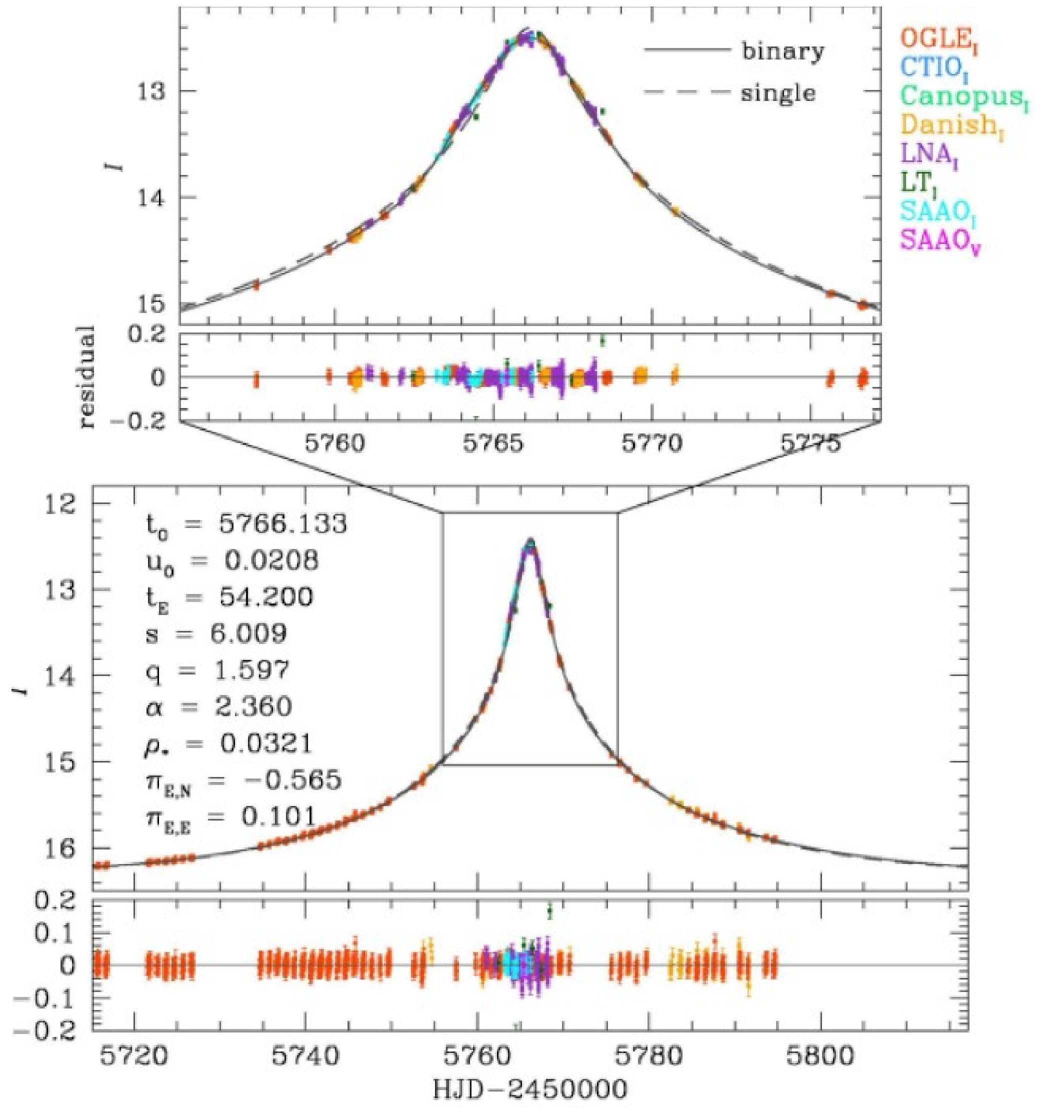




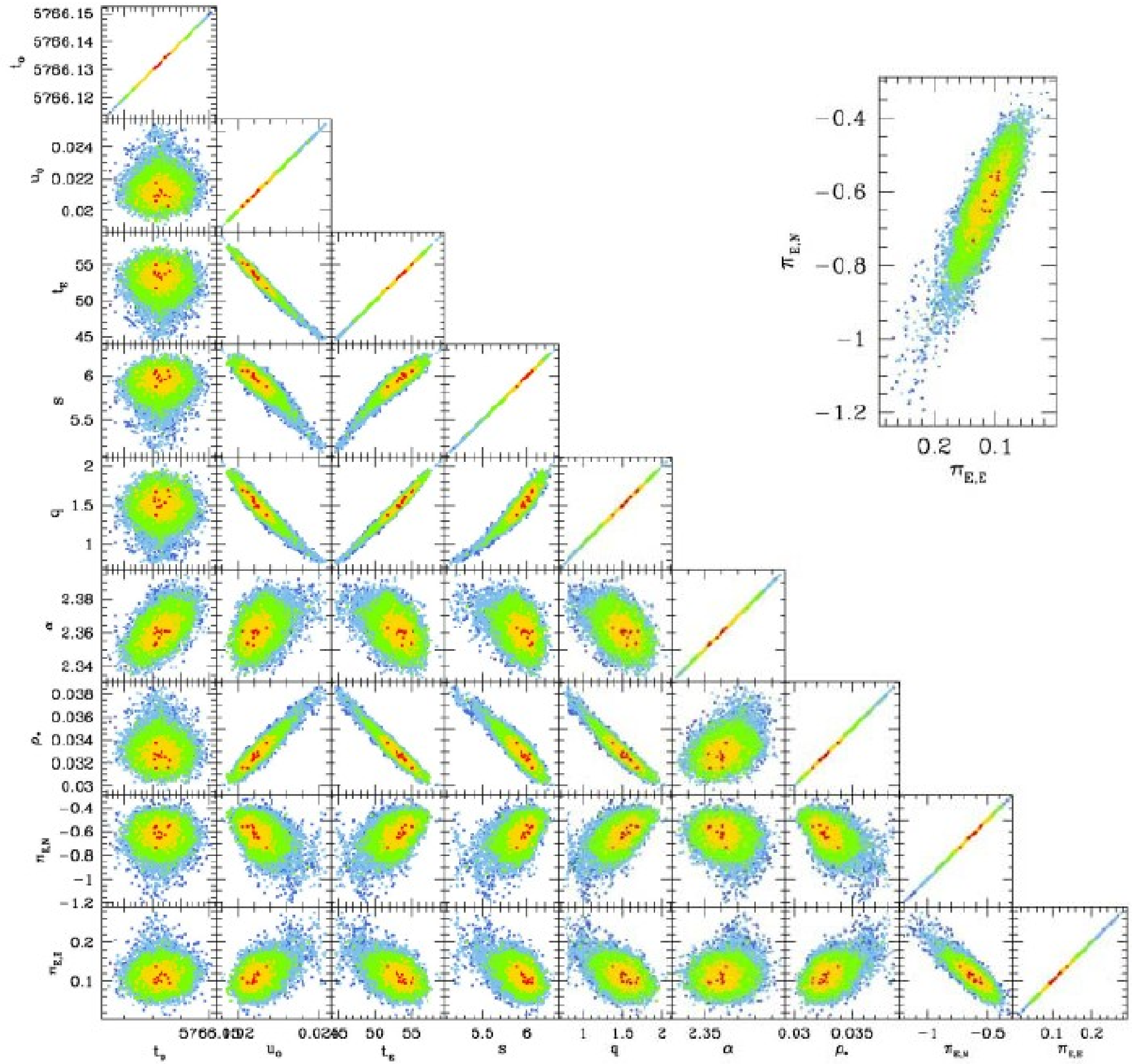
OGLE-2011-BLG-0420

Parallax + Orbital Motion





OGLE-2011-BLG-0420



OGLE-2011-BLG-0420

parameter	close	
	$u_0 > 0$	$u_0 < 0$
χ^2/dof	5427.4	5410.8
t_0 (HJD')	5766.110	5766.109
u_0	0.031	-0.030
t_E (days)	34.89	35.27
s	0.287	0.290
q	0.388	0.368
α	2.387	-2.383
$\rho_\star$	0.049	0.049
$\pi_{E,N}$	-1.03	-1.15
$\pi_{E,E}$	0.23	0.19
ds/dt (yr^{-1})	-2.44	-2.48
$d\alpha/dt$ (yr^{-1})	-8.09	7.08
KE/PE	0.36	0.32

quantity	close ($u_0 < 0$)
M_1	$0.024 \pm 0.001\ M_\odot$
M_2	$0.0088 \pm 0.0005\ M_\odot$ ($9.3 \pm 0.5\ M_J$)
D_L (kpc)	2.1 ± 0.1
projected separation (AU)	0.19 ± 0.01

(KE/PE)_perp: Ratio of Transverse Kinetic to Potential Energy

$$\mathrm{KE} = \frac{M_1 M_2}{M_1 + M_2}\frac{v_{\rm rel}^2}{2}; \quad \mathrm{PE} = \frac{G M_1 M_2}{r}$$

$$(\mathrm{KE})_\perp \equiv \frac{M_1 M_2}{M_1 + M_2}\frac{v_\perp^2}{2}; \quad (\mathrm{PE})_\perp \equiv \frac{G M_1 M_2}{r_\perp}$$

$$\left(\frac{\mathrm{KE}}{\mathrm{PE}}\right)_\perp = \left(\frac{\mathrm{KE}}{\mathrm{PE}}\right)\left(\frac{v_{\rm rel}}{v_\perp}\right)^2 \frac{r_\perp}{r} \le \left(\frac{\mathrm{KE}}{\mathrm{PE}}\right)$$

$$\left(\frac{\mathrm{KE}}{\mathrm{PE}}\right)_\perp = \frac{r_\perp v_{\rm rel}^2}{2GM} = \frac{r_\perp^3 \gamma^2}{2GM}$$

$$r_\perp = D_{\rm L}\theta_{\rm E} s = \frac{\mathrm{AU}\,\theta_{\rm E} s}{\pi_{\rm E}\theta_{\rm E} + \pi_s} = \frac{\mathrm{AU}\, s}{\pi_{\rm E} + \pi_s/\theta_{\rm E}}$$

$$\frac{\mathrm{AU}^3}{GM_\odot} = \left(\frac{\mathrm{yr}}{2\pi}\right)^2; \quad \frac{M}{M_\odot} = \frac{\theta_{\rm E}}{\kappa M_\odot \pi_{\rm E}} = \frac{\theta_{\rm E}/8.14\,\mathrm{mas}}{\pi_{\rm E}}$$

$$\left(\frac{\mathrm{KE}}{\mathrm{PE}}\right)_\perp = \frac{8.14}{8\pi^2}\frac{\pi_{\rm E} s^3 (\gamma\,\mathrm{yr})^2}{(\theta_{\rm E}/\mathrm{mas})(\pi_E + \pi_s/\theta_E)^3}$$

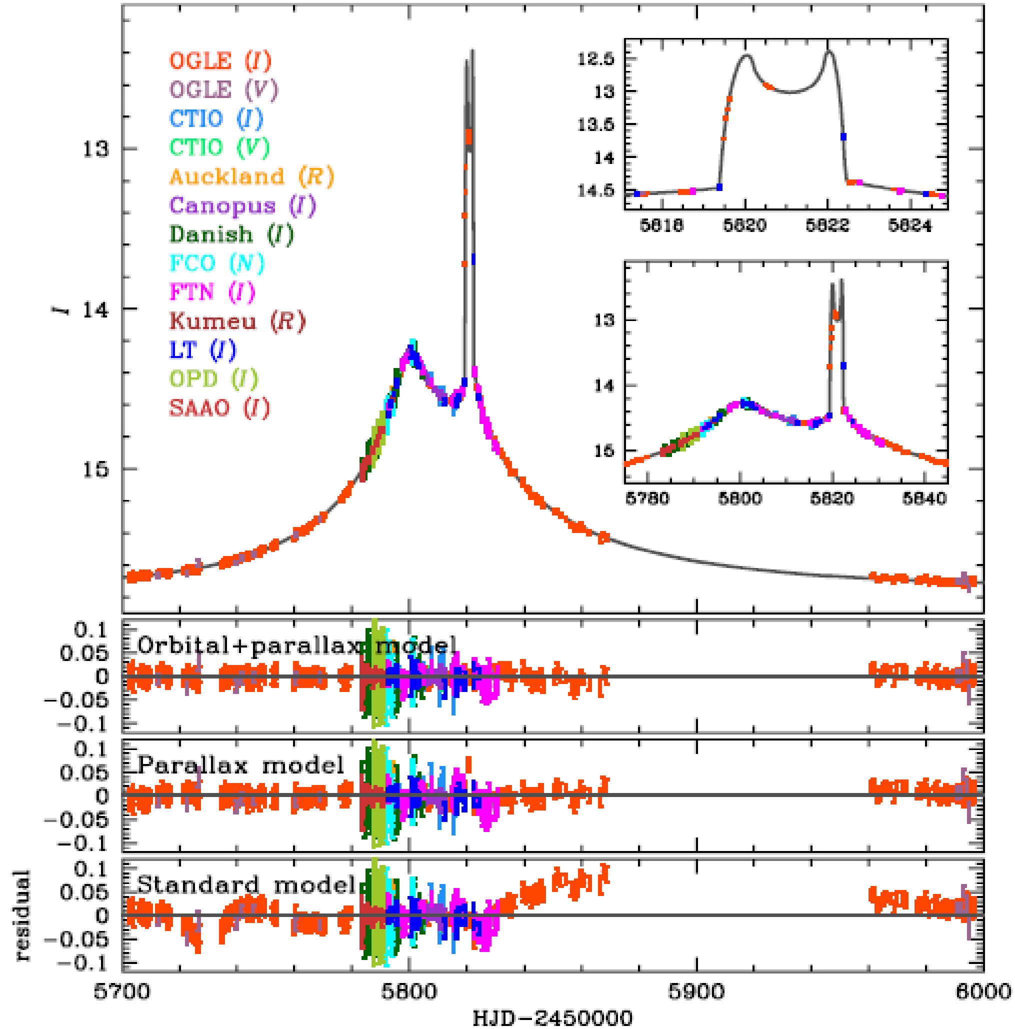
Complete Orbital Motion

13 “Features” & 13 Parameters

3 Point-Lens	t_0, u_0, t_E
3 Binary-Lens	α_0, s_0, q
Width of Caustic Cr.	$t_* = \rho * t_E$
2 Parallax	$\pi_{E,perp}$, $\pi_{E,parallel}$
2 Transverse Motion	γ_{perp}, $\gamma_{parallel}$
Out-of-plane Position	$s_{parallel}$
Out-of-plane Motion	$ds_{parallel}/dt$

OGLE-2011-BLG-0417

Complete Orbital Solution



Shin et al. 2012, ApJ, 755, 91

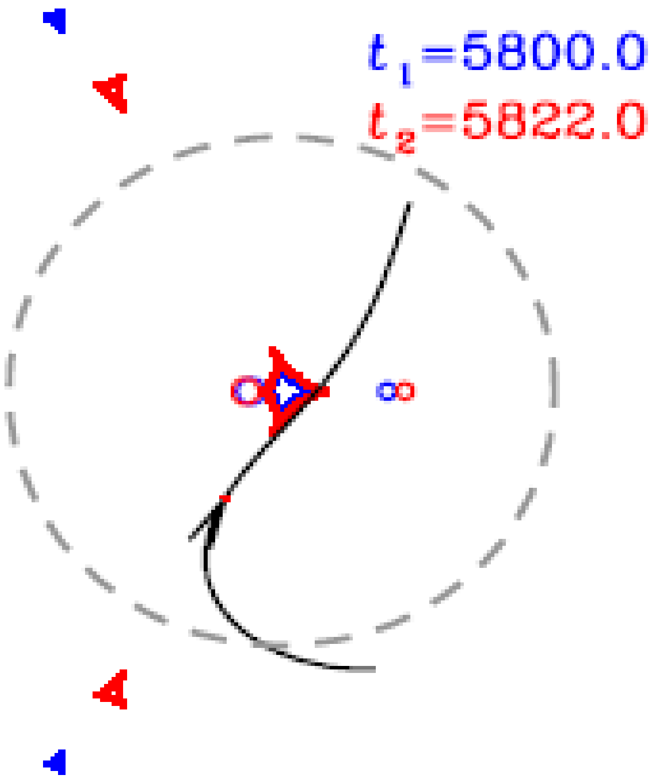
OGLE-2011-BLG-0417

Complete Orbital Solution

Parameters	Standard	Model Parallax	Orbital+Parallax
χ^2/dof	4415/2627	2391/2625	1735/2621
t_0 (HJD′)	5817.302 ± 0.018	5815.867 ± 0.030	5813.306 ± 0.059
u_0	0.1125 ± 0.0001	-0.0971 ± 0.0003	-0.0992 ± 0.0005
t_E (days)	60.74 ± 0.08	79.59 ± 0.36	92.26 ± 0.37
$s_\perp$	0.601 ± 0.001	0.574 ± 0.001	0.577 ± 0.001
q	0.402 ± 0.002	0.287 ± 0.002	0.292 ± 0.002
α (rad)	1.030 ± 0.002	-0.951 ± 0.002	-0.850 ± 0.004
ρ_* (10^{-3})	3.17 ± 0.01	2.38 ± 0.02	2.29 ± 0.02
$\pi_{E,N}$	...	0.125 ± 0.004	0.375 ± 0.015
$\pi_{E,E}$	...	-0.111 ± 0.005	-0.133 ± 0.003
$ds_\perp/dt$ (yr^{-1})	...	...	1.314 ± 0.023
$d\alpha/dt$ (yr^{-1})	...	...	1.168 ± 0.076
$s_\parallel$	...	...	0.467 ± 0.020
$ds_\parallel/dt$ (yr^{-1})	...	...	-0.192 ± 0.036

OGLE-2011-BLG-0417

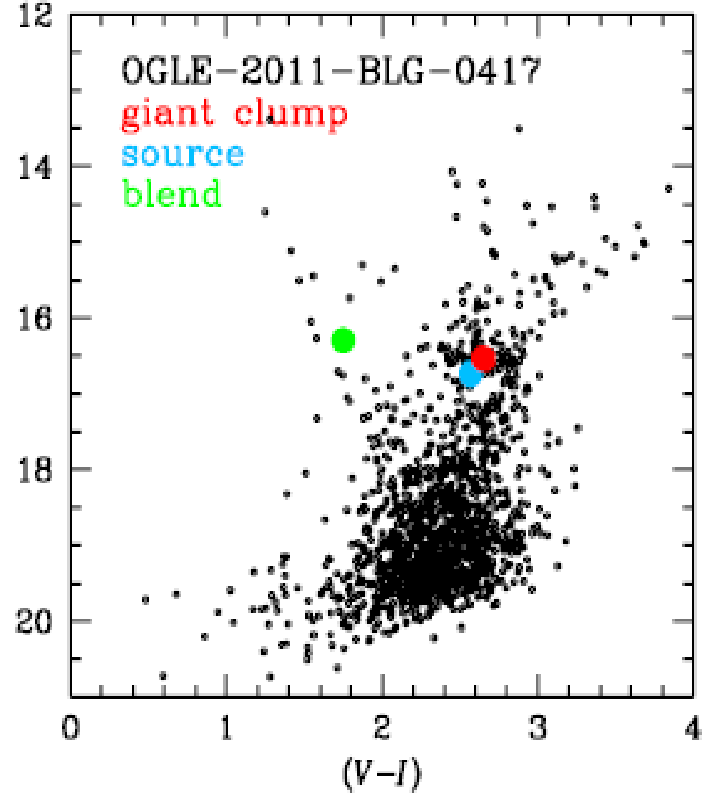
Complete Orbital Solution



Parameter	OGLE-2011-BLG-0417
M_{tot} ($M_\odot$)	0.74 ± 0.03
M_1 ($M_\odot$)	0.57 ± 0.02
M_2 ($M_\odot$)	0.17 ± 0.01
θ_E (mas)	2.44 ± 0.02
μ (mas yr^{-1})	9.66 ± 0.07
D_L (kpc)	0.89 ± 0.03
a (AU)	1.15 ± 0.04
P (yr)	1.44 ± 0.06
e	0.68 ± 0.02
i (deg)	116.95 ± 1.04

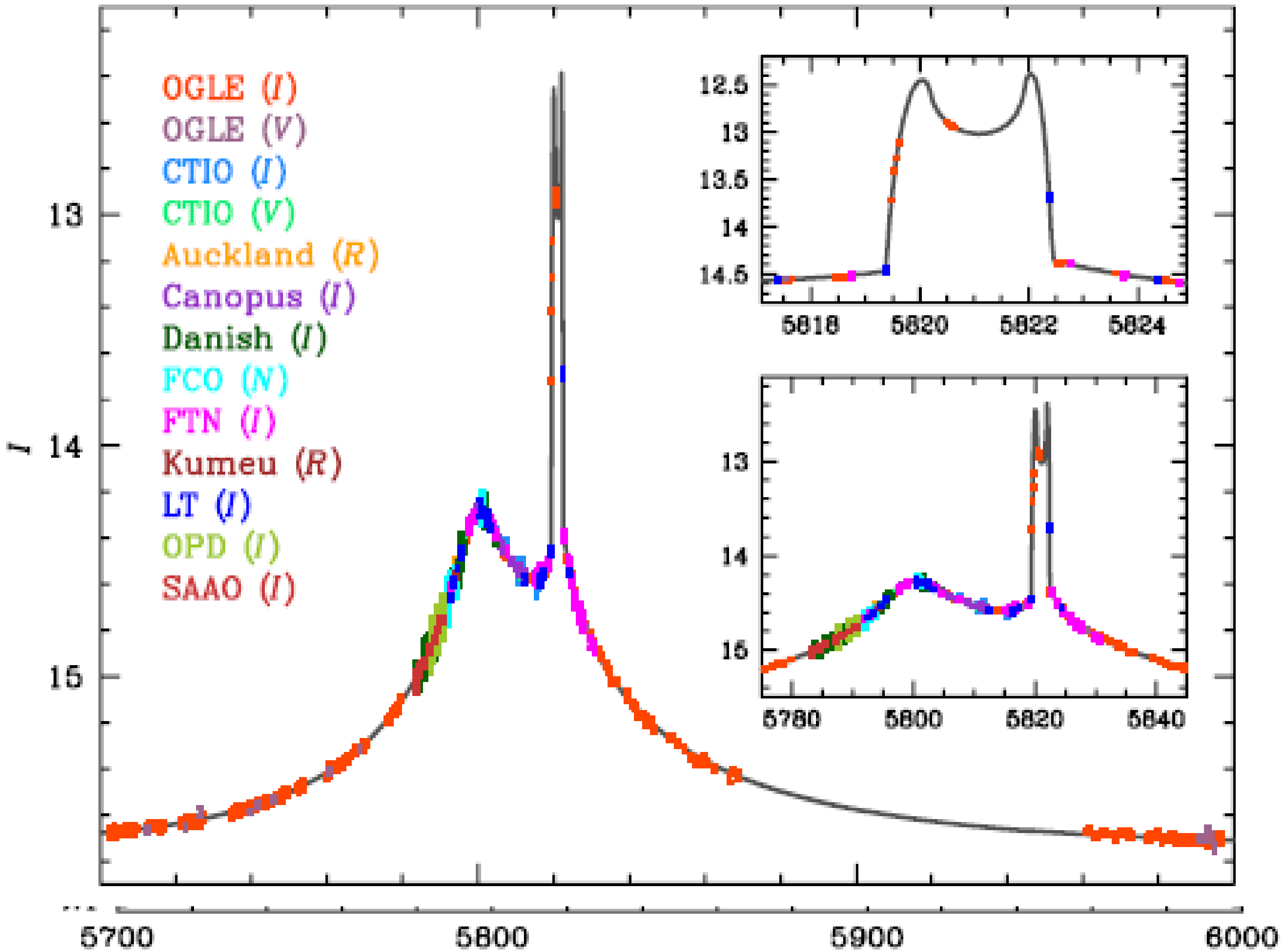
OGLE-2011-BLG-0417

Complete Orbital Solution



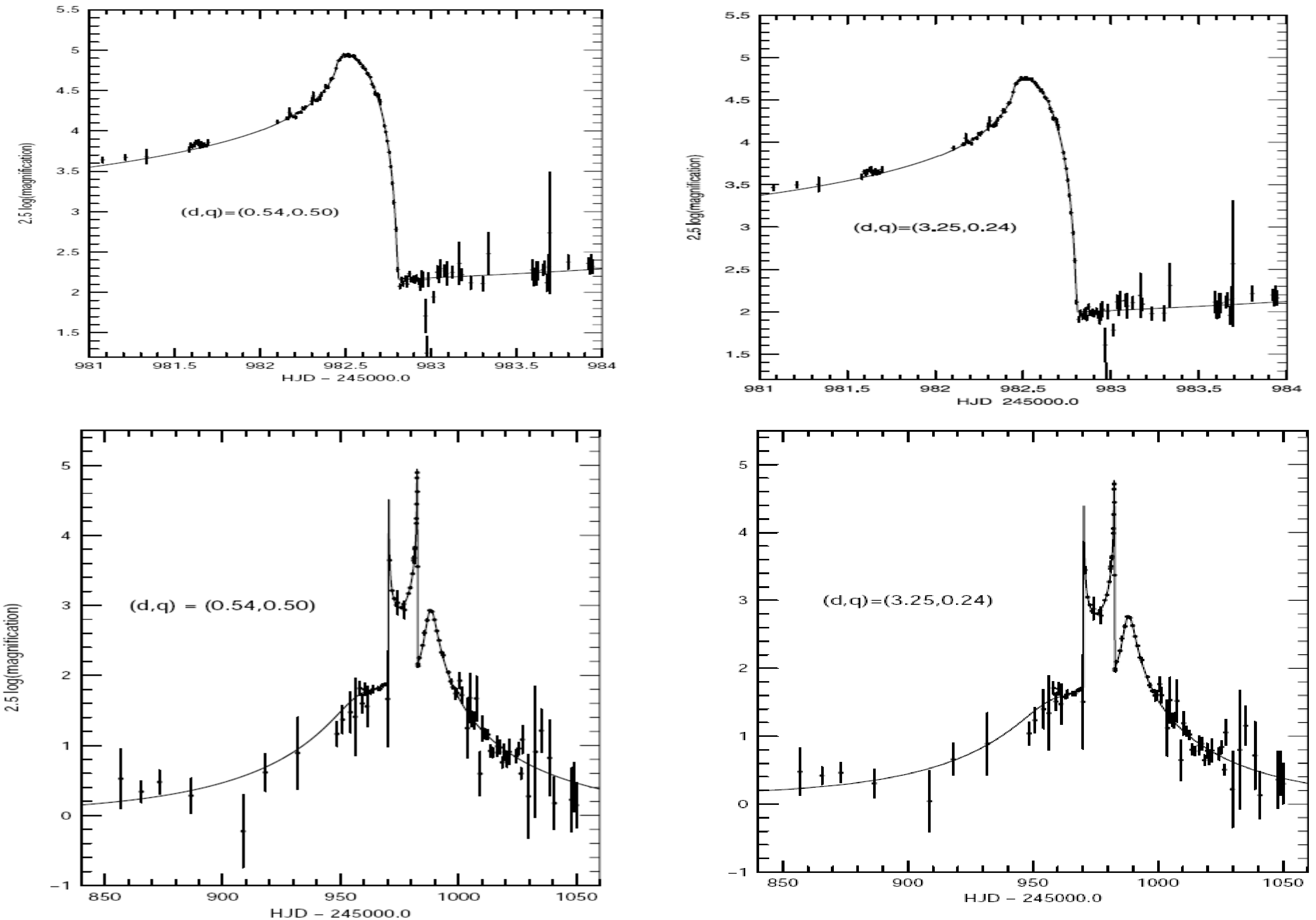
Parameter	OGLE-2011-BLG-0417
M_{tot} ($M_\odot$)	0.74 ± 0.03
M_1 ($M_\odot$)	0.57 ± 0.02
M_2 ($M_\odot$)	0.17 ± 0.01
θ_E (mas)	2.44 ± 0.02
μ (mas yr^{-1})	9.66 ± 0.07
D_L (kpc)	0.89 ± 0.03
a (AU)	1.15 ± 0.04
P (yr)	1.44 ± 0.06
e	0.68 ± 0.02
i (deg)	116.95 ± 1.04

OGLE-2011-BLG-0417

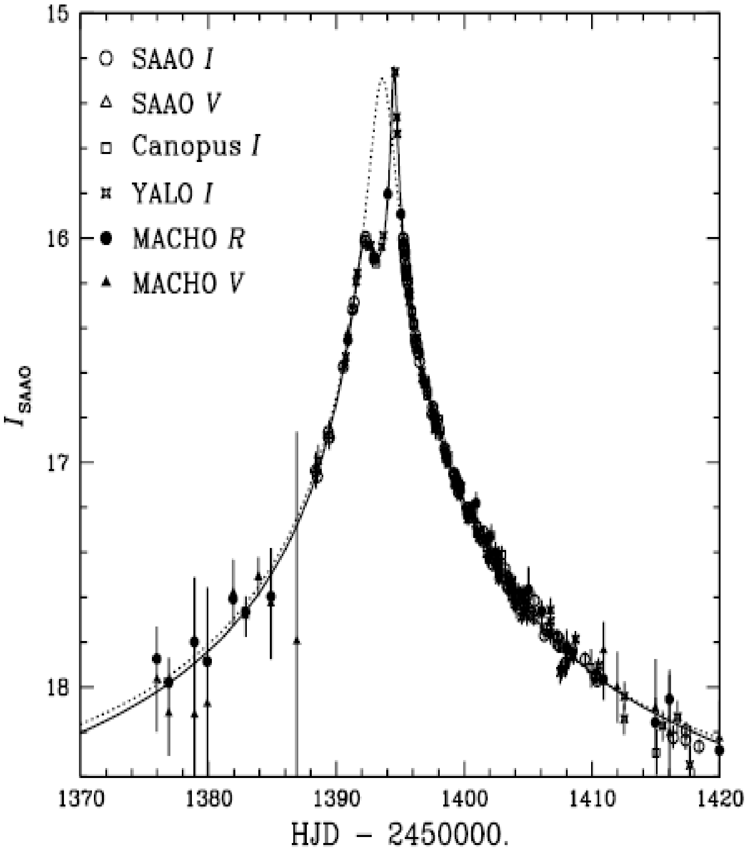


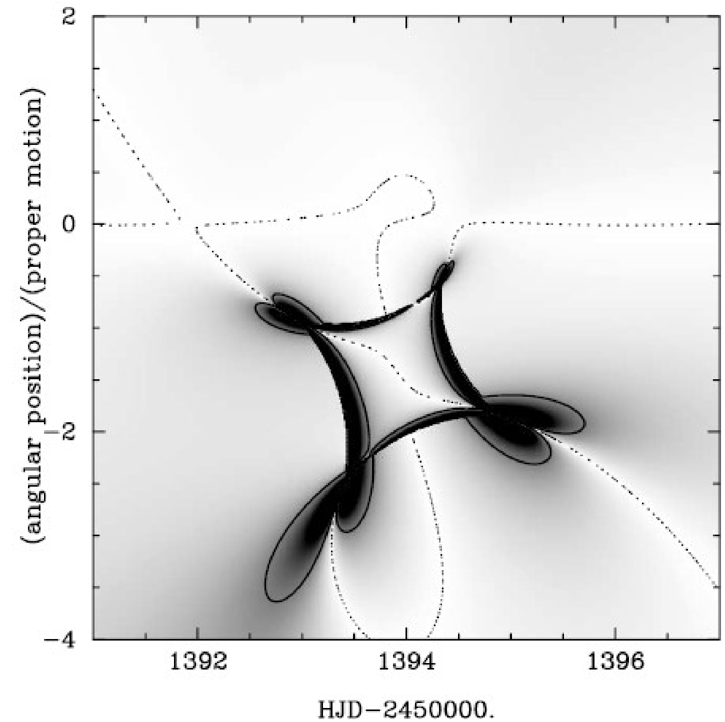
Macho-98-SMC-1

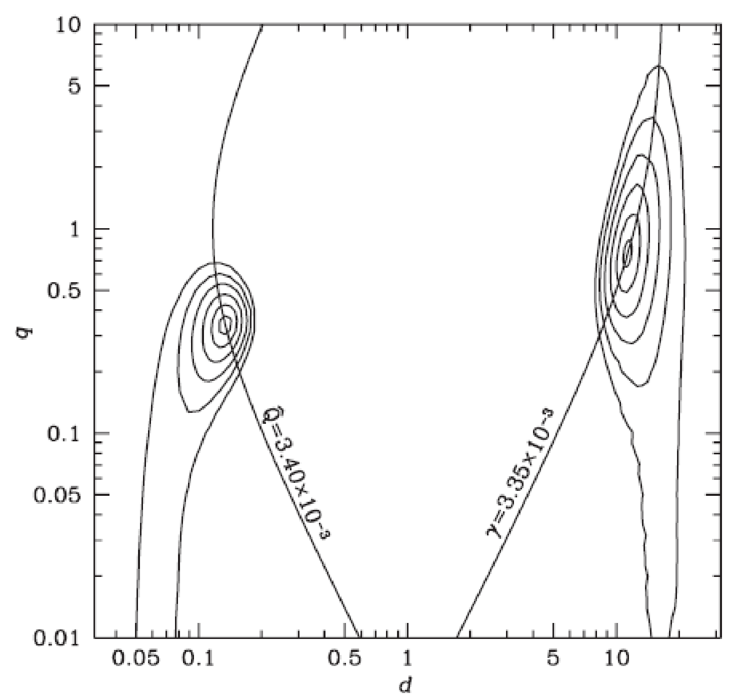
Close/Wide Binary Degeneracy



Macho-99-BLG-47







Jin An: Close/Wide Degeneracy (At Lowest Order) [d & q]

$$\zeta = z - \frac{\epsilon_1}{\bar{z} - d_c\epsilon_2} - \frac{\epsilon_2}{\bar{z} + d_c\epsilon_1}$$
$$\approx z - \frac{1}{\bar{z}} - \frac{d_c^2\epsilon_1\epsilon_2}{\bar{z}^3} + \frac{d_c^3\epsilon_1\epsilon_2(\epsilon_1 - \epsilon_2)}{\bar{z}^4} + \cdots$$

$$\delta z_c \approx \hat{Q}\left(1 - \frac{1}{|z_0|^4}\right)^{-1}\left[\left(\frac{1}{\bar{z}_0^3} - \frac{1}{z_0^3\bar{z}_0^2}\right) + \left(\frac{1}{z_0^4\bar{z}_0^2} - \frac{1}{\bar{z}_0^4}\right)\frac{1-q_c}{1+q_c}d_c + \cdots\right]$$

$$\frac{\partial\zeta}{\partial\bar{z}} \approx \frac{1}{\bar{z}^2} + \frac{3\hat{Q}}{\bar{z}^4}\left[1 - \frac{4(1-q_c)}{3(1+q_c)}\frac{d_c}{\bar{z}} + \cdots\right]$$

$$\left|\frac{\partial\zeta}{\partial\bar{z}}\right|^2 \approx \frac{1}{|z_0|^4} + \hat{Q}\left[\frac{3|z_0|^4 - 2|z_0|^2 - 1}{|z_0|^8 - |z_0|^4}\left(\frac{1}{z_0^2} + \frac{1}{\bar{z}_0^2}\right) - \frac{4|z_0|^4 - 2|z_0|^2 - 2}{|z_0|^8 - |z_0|^4}\left(\frac{1}{z_0^3} + \frac{1}{\bar{z}_0^3}\right)\frac{1-q_c}{1+q_c}d_c + \cdots\right]$$

$$A^{-1} \approx \left|4\Delta - 2\hat{Q}\left(\frac{1}{z_0^2} + \frac{1}{\bar{z}_0^2}\right) + 3\hat{Q}\left(\frac{1}{z_0^3} + \frac{1}{\bar{z}_0^3}\right)\frac{1-q_c}{1+q_c}d_c\right|$$
$$= 4\left|(|z_0| - 1) - \hat{Q}\Re(z_0^{-2}) + \frac{3(1-q_c)}{2(1+q_c)}d_c\hat{Q}\Re(z_0^{-3})\right|$$

$$\zeta = z - \frac{1}{\bar{z}} - \frac{q_w}{\bar{z} + d_1},$$
$$\approx z - \frac{1}{\bar{z}} - \frac{q_w}{d_1} + \frac{q_w}{d_1^2}\bar{z} - \frac{q_w}{d_1^3}\bar{z}^2 + \cdots \qquad (d_w \gg |z|)$$

$$\delta z_w \approx \gamma\left(1 - \frac{1}{|z_0|^4}\right)^{-1}\left[\left(\frac{z_0}{\bar{z}_0^2} - \bar{z}_0\right) + \left(\bar{z}_0^2 - \frac{z_0^2}{\bar{z}_0^2}\right)\frac{1}{(1+q_w)^{1/2}d_w} + \cdots\right]$$

$$\frac{\partial\zeta}{\partial\bar{z}} \approx \frac{1}{\bar{z}^2} + \gamma\left[1 - \frac{2}{(1+q_w)^{1/2}}\frac{\bar{z}}{d_w} + \cdots\right]$$

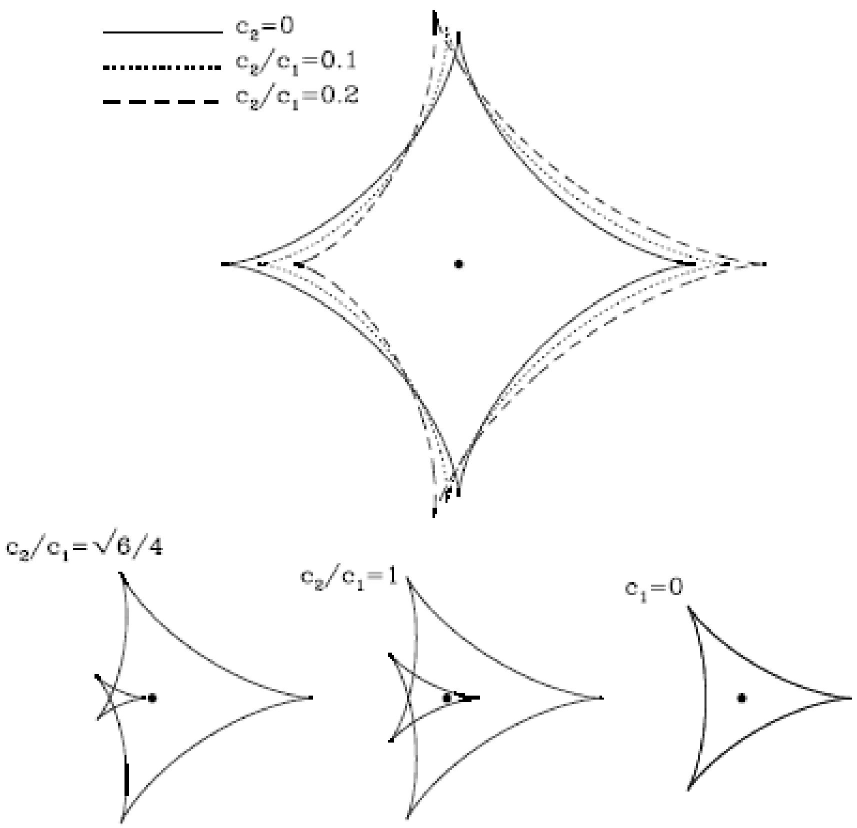
$$\left|\frac{\partial\zeta}{\partial\bar{z}}\right|^2 \approx \frac{1}{|z_0|^4} + \gamma\left[\frac{|z_0|^4 + 2|z_0|^2 - 3}{|z_0|^4 - 1}\left(\frac{1}{z_0^2} + \frac{1}{\bar{z}_0^2}\right) - \frac{2|z_0|^6 + 2|z_0|^4 - 4|z_0|^2}{|z_0|^4 - 1}\left(\frac{1}{z_0^3} + \frac{1}{\bar{z}_0^3}\right)\frac{1}{(1+q_w)^{1/2}d_w} + \cdots\right]$$

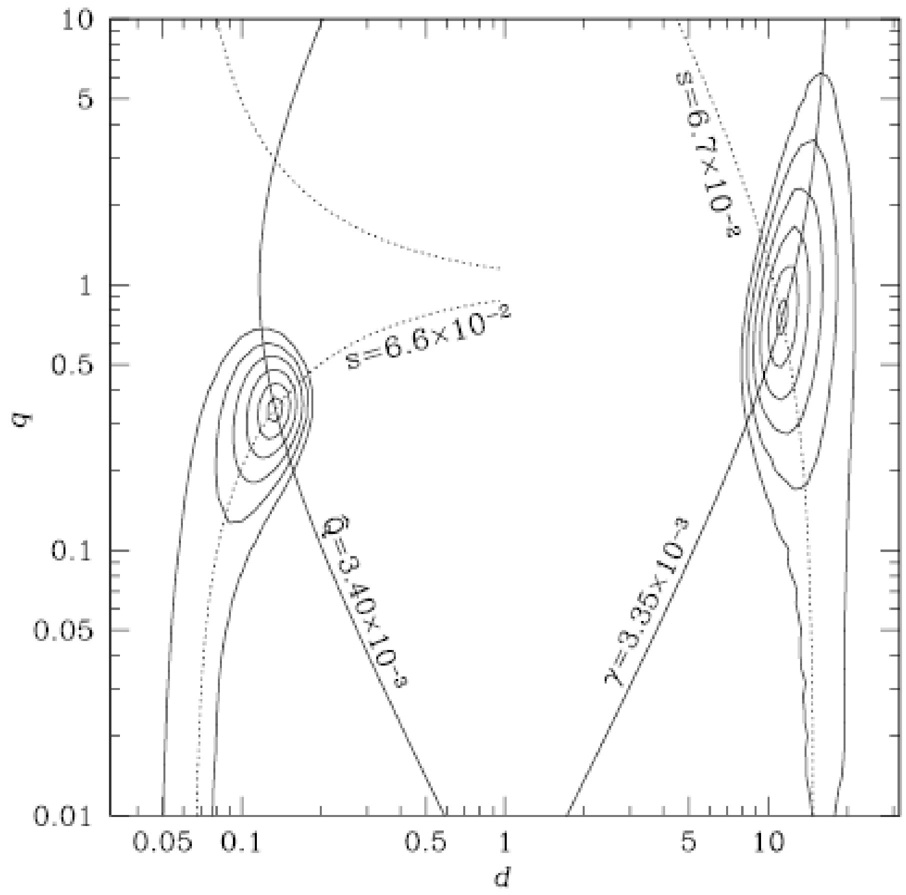
$$A^{-1} \approx \left|4\Delta - 2\gamma\left(\frac{1}{z_0^2} + \frac{1}{\bar{z}_0^2}\right) + 3\gamma\left(\frac{1}{z_0^3} + \frac{1}{\bar{z}_0^3}\right)\frac{1}{(1+q_w)^{1/2}d_w}\right|$$
$$= 4\left|(|z_0| - 1) - \gamma\Re(z_0^{-2}) + \frac{3}{2(1+q_w)^{1/2}}\frac{\gamma}{d_w}\Re(z_0^{-3})\right|$$

$$d_c^2 d_w^2(1+q_w) = \frac{q_w}{q_c}(1+q_c)^2,$$

$$d_c d_w(1+q_w)^{1/2} = \frac{1+q_c}{1-q_c},$$

Jin An: Wide/Close Degeneracy (At Second Order) [Shape Parameter: s = c_2/c_1]



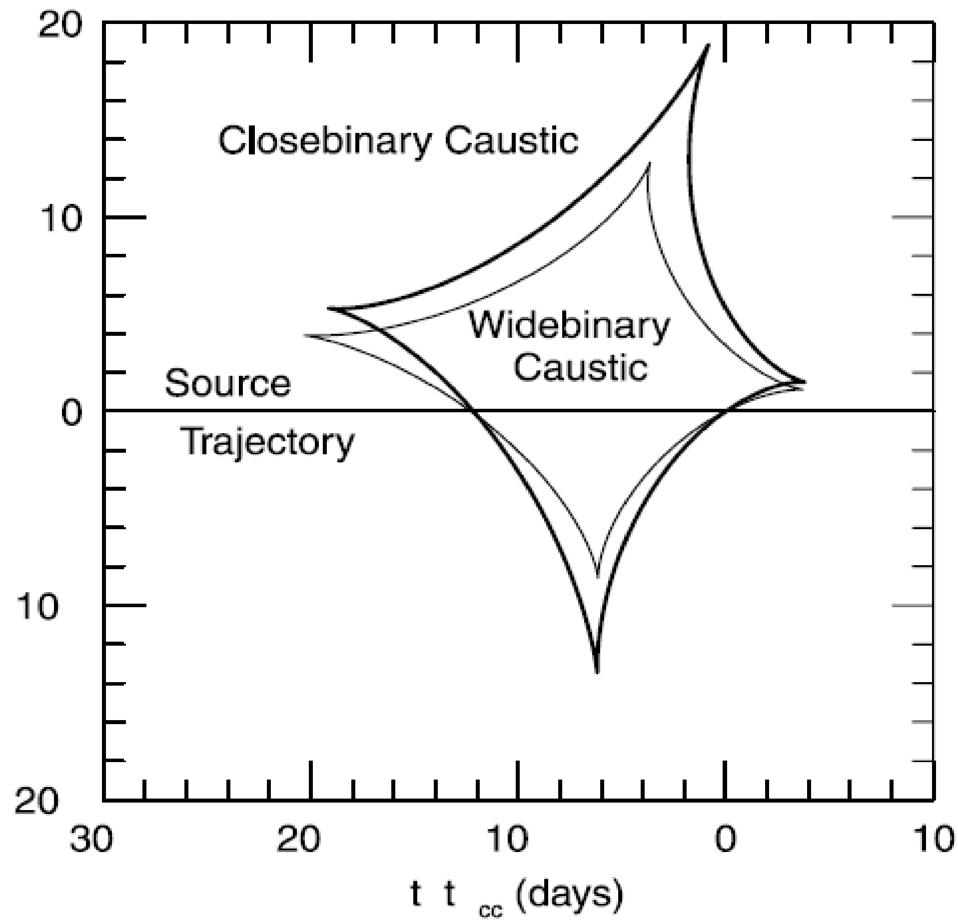


Different caustics -> Same lightcurve

An 2005, MNRAS, 356, 1409

$$q_c \to 1 - \frac{\sqrt{1+4q_w} - 1}{2q_w}$$

$$b_c \to b_w^{-1} \sqrt{\frac{1+4q_w}{1+q_w}}$$

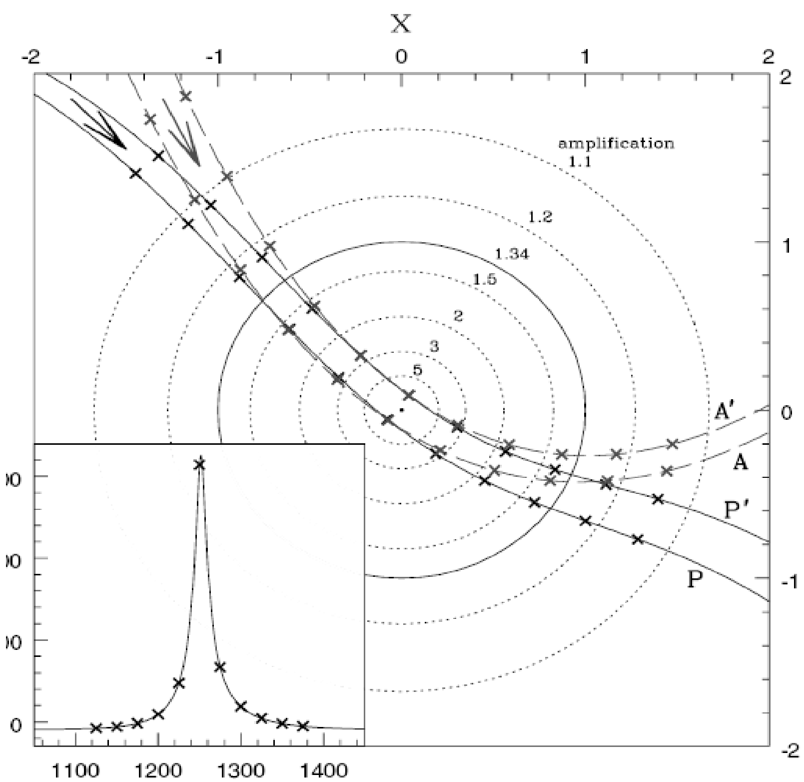


Ecliptic Degeneracy

Begins in 'constant acceleration' model

u_0 --> -u_0

Smith, Mao, & Paczynski (2003)



Ecliptic Degeneracy

Embedded in 'jerk parallax' formalism

u_0 --> -u_0

|u_0|<<1 ==> jerk-par

SMP (2003)

Gould (2004)

$$\pi'_{E,\parallel} = \pi_{E,\parallel}, \qquad \pi'_{E,\perp} = -(\pi_{E,\perp} + \pi_{j,\perp}),$$

$$\pi_{j,\perp} = -\frac{4}{3}\frac{\mathrm{yr}}{2\pi t_E}\frac{\sin\beta_{ec}}{(\cos^2\psi\sin^2\beta_{ec} + \sin^2\psi)^{3/2}}$$

Ecliptic Degeneracy

Jiang et al.: Exact Degenercy (β_{ec}=0)

u_0 --> -u_0

|u_0|<<1 ==> jerk-par

$(u_0,\pi_{E,perp})$->-$(u_0,\pi_{E,perp})$

SMP (2003)

Gould (2004)

Jiang et al. (2004)

$$\pi_{j,\perp} = -\frac{4}{3}\frac{\mathrm{yr}}{2\pi t_{\mathrm{E}}}\frac{\sin\beta_{\mathrm{ec}}}{\left(\cos^2\psi\sin^2\beta_{\mathrm{ec}} + \sin^2\psi\right)^{3/2}}$$

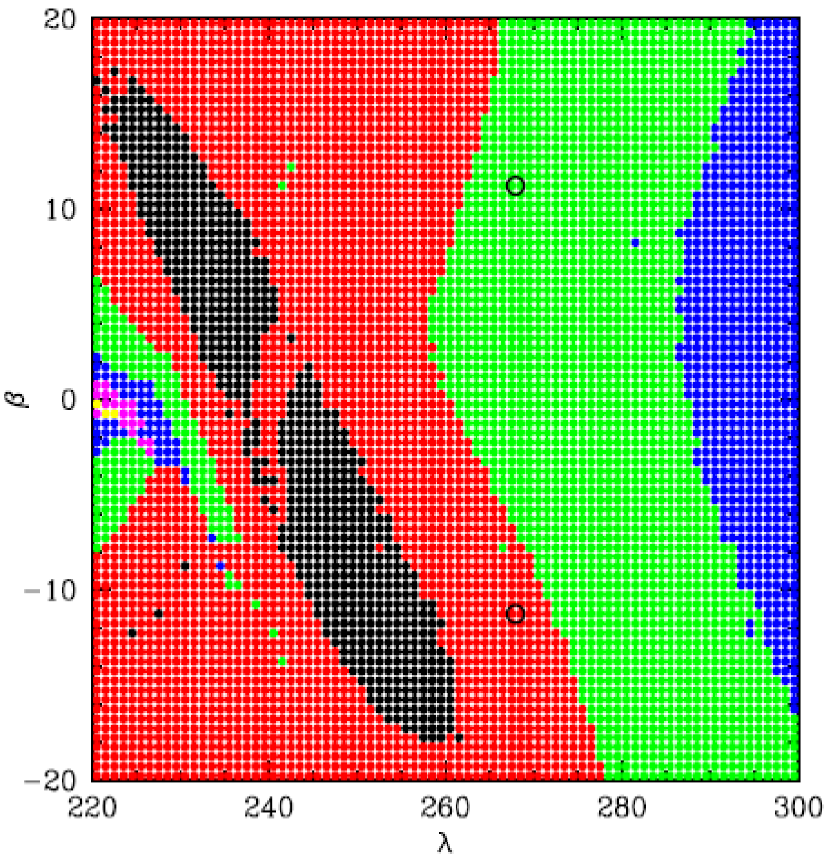
Ecliptic Degeneracy

Skowron et al. 2011, ApJ, 738,87

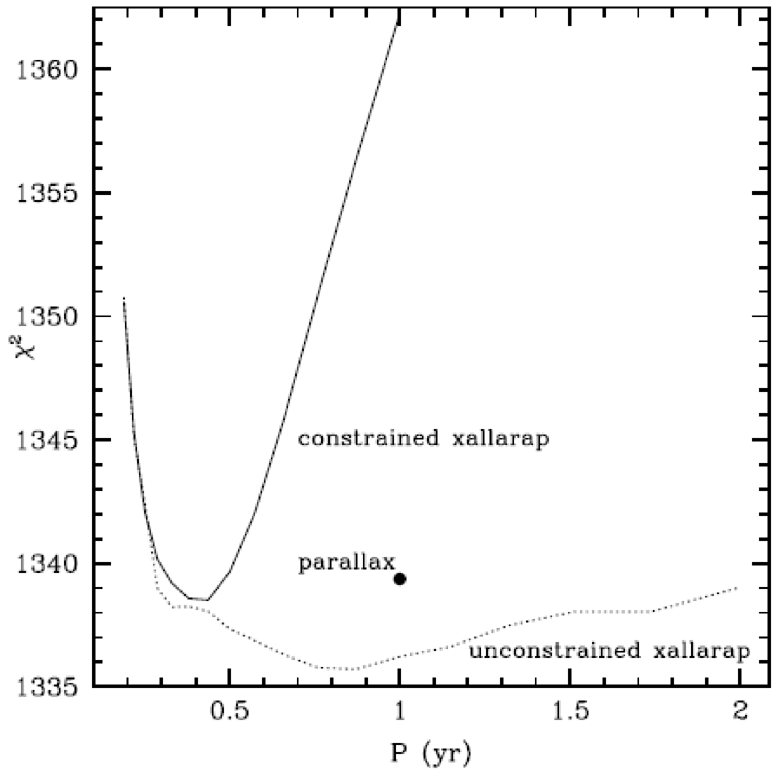
generalize to binaries

u_0 --> -u_0	SMP (2003)
|u_0|<<1 ==> jerk-par	Gould (2004)
$(u_0,\pi_{E,perp})$-> $-(u_0,\pi_{E,perp})$	Single
$(u_0,\pi_{E,perp},\alpha)$ -> $-(u_0,\pi_{E,perp},\alpha)$	Static Binary
$(u_0,\pi_{E,perp},\alpha_0,d\alpha/dt)$ -> $-(u_0,\pi_{E,perp},\alpha_0,d\alpha/dt)$	Rotating Binary

Xallarap vs. Parallax



Xallarap vs. Parallax



Point-lens magnfication

Start: Binary-Lens Equation

$$\mathbf{u} - \mathbf{y} = -\frac{\mathbf{y} - \mathbf{y}_L}{|\mathbf{y} - \mathbf{y}_L|^2}$$

$$\mathbf{y}_L = 0 \rightarrow \mathbf{u} - \mathbf{y} = -\frac{\mathbf{y}}{y^2} \Longrightarrow u - y = -\frac{1}{y}$$

$$\Longrightarrow (y - u)y = 1 \Longrightarrow (\theta_I - \theta_S)\theta_I = \theta_{\mathrm{E}}^2$$

$$\mathbf{u} = \mathbf{y} - \sum_i \epsilon_i \frac{\mathbf{y} - \mathbf{y}_{m,i}}{|\mathbf{y} - \mathbf{y}_{m,i}|^2} \qquad \epsilon_i \equiv \frac{m_i}{M_{\mathrm{tot}}}$$

$$\zeta = z - \sum_i \frac{\epsilon_i}{\bar{z} - \bar{z}_{m,i}}$$

$$\zeta \equiv u_1 + i u_2 \qquad z \equiv y_1 + i y_2$$

Why is this a Fifth-Order Equation?

$$\zeta = z - \sum_i \frac{\epsilon_i}{\bar{z} - \bar{z}_{m,i}}$$

$$\zeta \equiv u_1 + iu_2; \qquad z \equiv y_1 + iy_2$$

$$z = \zeta + \frac{\epsilon_1}{\bar{z} - \bar{z}_1} + \frac{\epsilon_2}{\bar{z} - \bar{z}_2}$$

$$\bar{z} = \bar{\zeta} + \frac{\epsilon_1}{z - z_1} + \frac{\epsilon_2}{z - z_2}$$

$$(z - \zeta)(\bar{z} - \bar{z}_1)(\bar{z} - \bar{z}_2) = \epsilon_1(\bar{z} - \bar{z}_2) + \epsilon_2(\bar{z} - \bar{z}_1)$$

$$(z - \zeta)\left(\bar{\zeta} + \frac{\epsilon_1}{z - z_1} + \frac{\epsilon_2}{z - z_2} - \bar{z}_1\right)\left(\bar{\zeta} + \frac{\epsilon_1}{z - z_1} + \frac{\epsilon_2}{z - z_2} - \bar{z}_2\right)$$

$$= \left(\bar{\zeta} + \frac{\epsilon_1}{z - z_1} + \frac{\epsilon_2}{z - z_2} - \bar{z}_2\right)\epsilon_1 + \left(\bar{\zeta} + \frac{\epsilon_1}{z - z_1} + \frac{\epsilon_2}{z - z_2} - \bar{z}_1\right)\epsilon_2$$

Magnification (A): For each image, i

1) Check that it solves lens equation

2) Calculate A_i from determinant

$$\partial\zeta_i = \sum_k \frac{\epsilon_k}{(\bar{z} - \bar{z}_k)^2}$$

$$A_i = \frac{1}{1 - |\partial\zeta_i|^2}$$

$$A = \sum_i |A_i|$$

Quadrupole/Hexadecapole

Pejcha & Heyrovsky (2009)

Gould (2008)

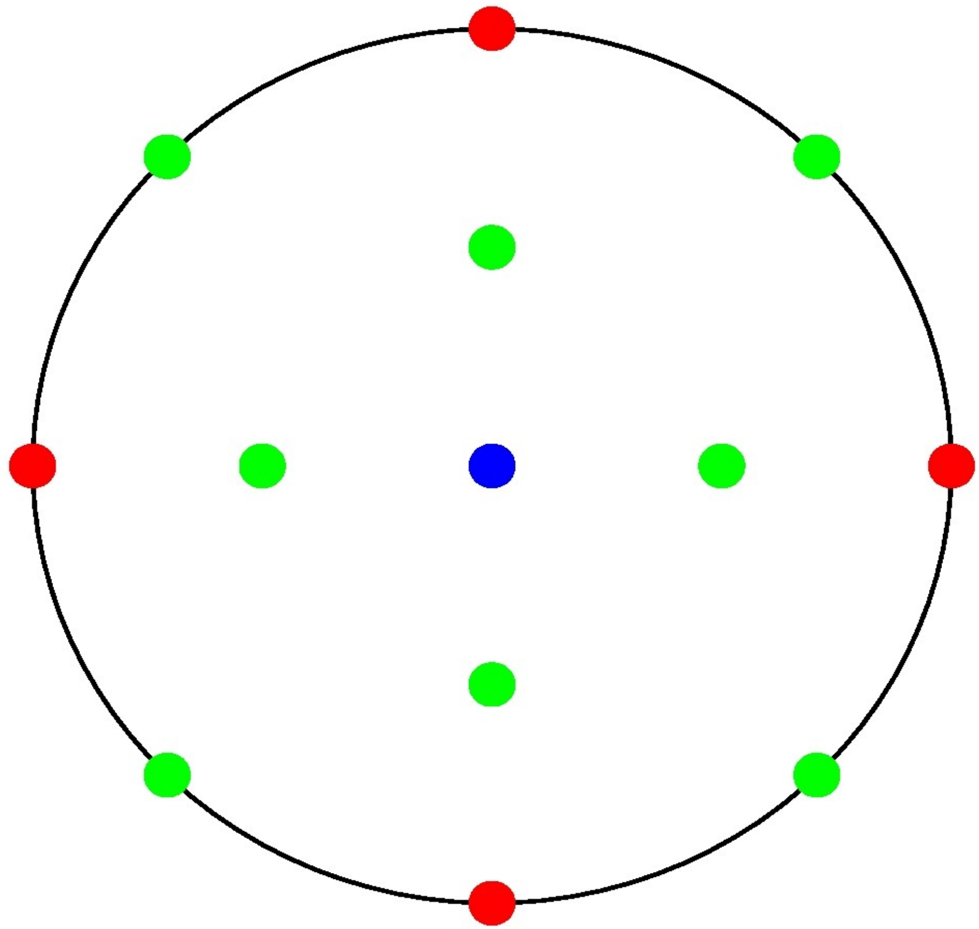
Pure gradient: Monopole (pt lens)

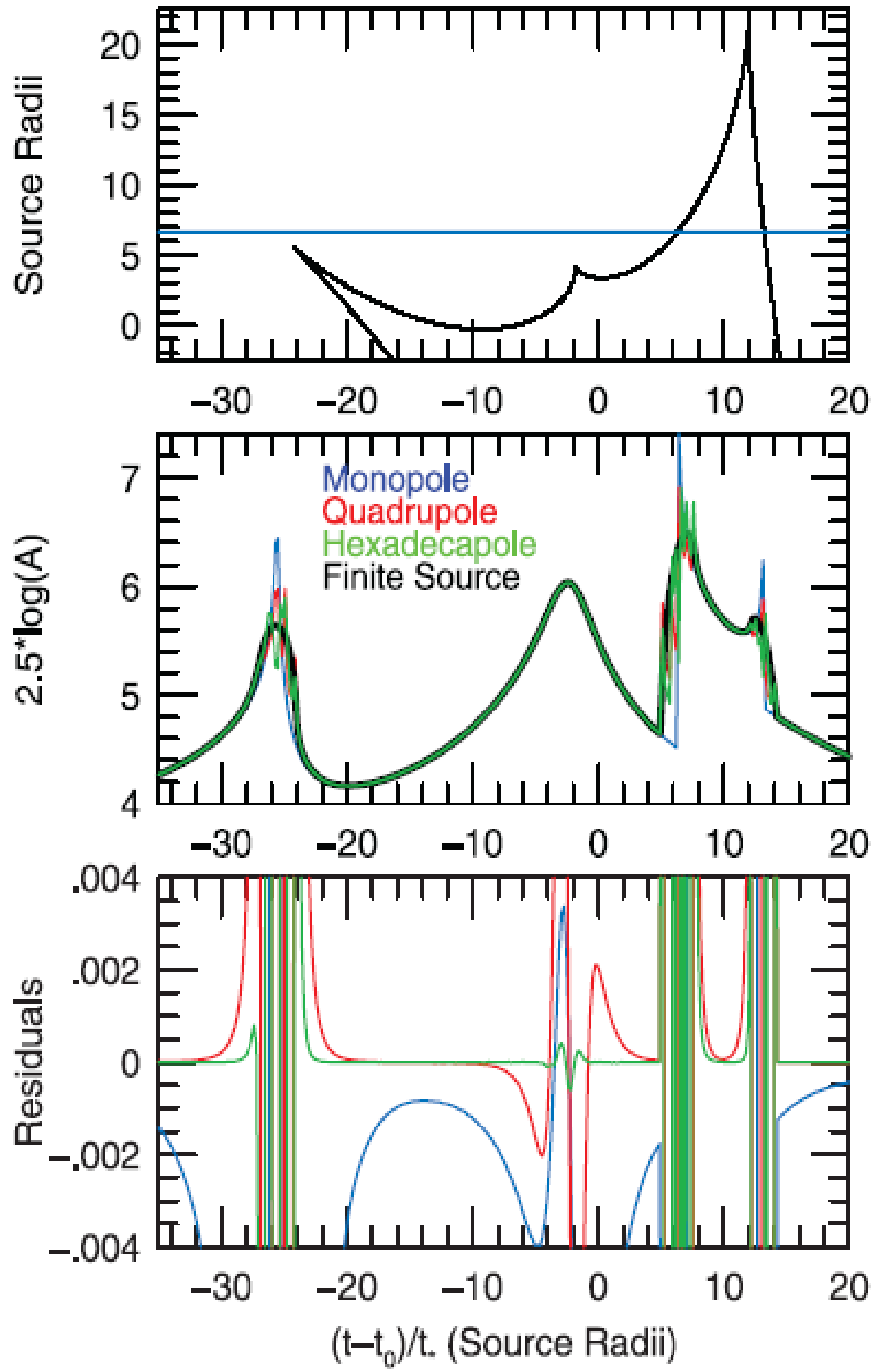
Mild curvature: Quadrupole

Stronger curvature: Hexadecapole

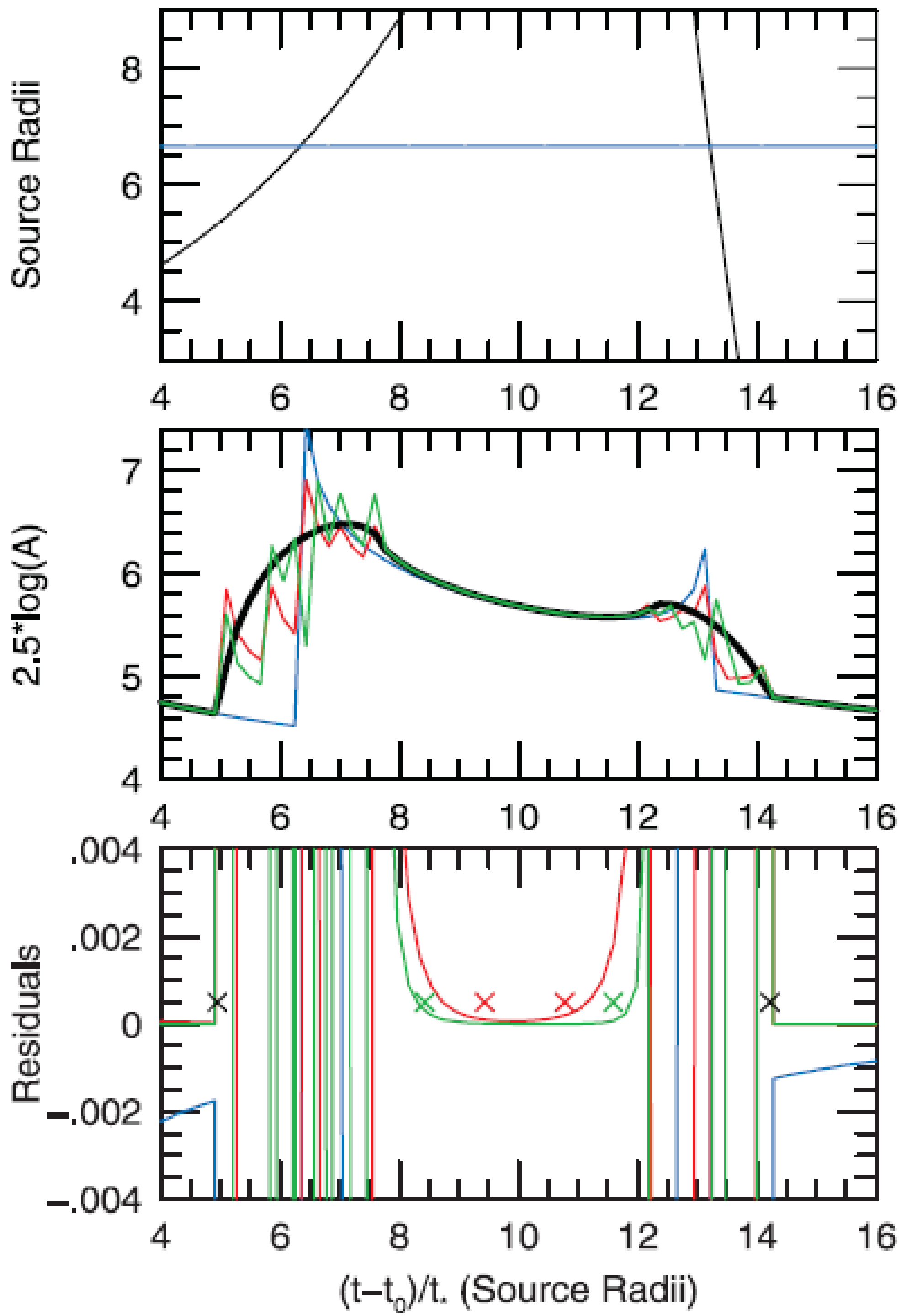
Extreme curvature: New Method

Monopole/Quadrupole/Hexadecapole

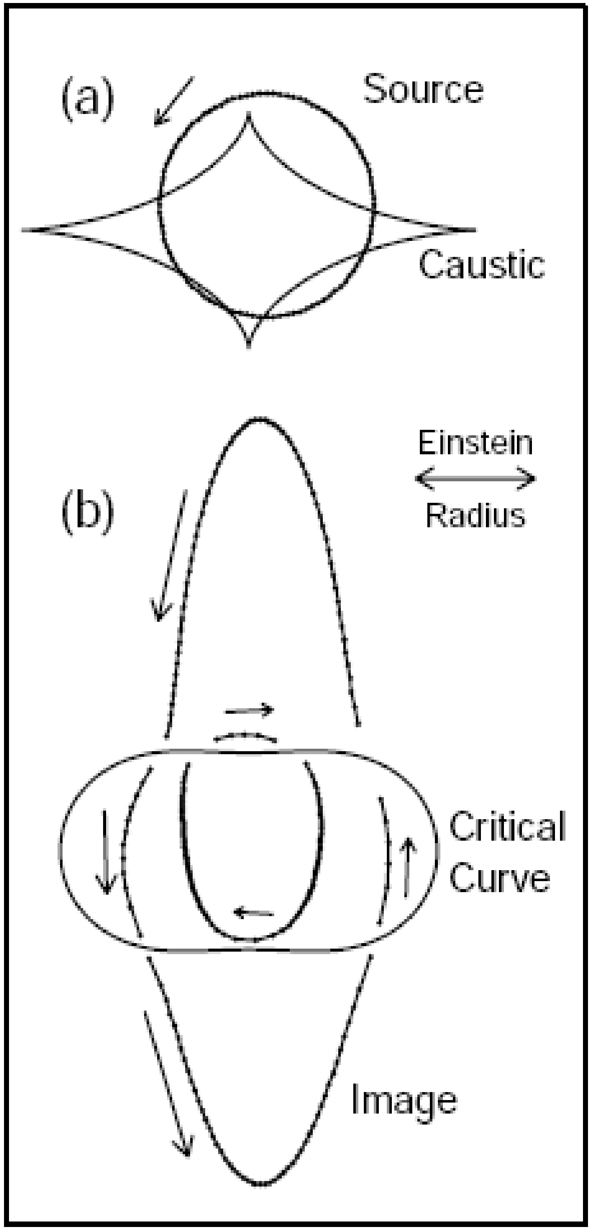




Source Radii
20
15
10
5
0
−30
−20
−10
0
10
20
Monopole
Quadrupole
Hexadecapole
Finite Source
2.5*log(A)
7
6
5
4
Residuals
.004
.002
0
−.002
−.004
(t−t0)/t. (Source Radii)


Source Radii
8
6
4
4
6
8
10
12
14
16
2.5*log(A)
7
6
5
4
4
6
8
10
12
14
16
Residuals
.004
.002
0
-.002
-.004
4
6
8
10
12
14
16
(t-t0)/t. (Source Radii)

Contour Integration: Gould & Gaucherel (1997)



$$A = \sum_{i=1}^{n} \sum_{j'} p_j (u_{i-1,j} \times u_{i,j}) \Big/ \sum_{i=1}^{n} s_{i-1} \times s_i \ ,$$

Contour Integration

Fastest INDIVIDUAL FS calculation

Disadvantage 1: Limb Darkening -> many cont.

Disadvantage 2: Cusp lens-solver hang-ups

Neither fatal (see below)

Major improvements from Bozza (2010)

MNRAS 408 2188 (code not public)

Inverse Ray Shooting (General)

$$\mathbf{u} = \mathbf{y} - \sum_i \epsilon_i \frac{\mathbf{y} - \mathbf{y}_{m,i}}{|\mathbf{y} - \mathbf{y}_{m,i}|^2} \qquad \epsilon_i \equiv \frac{m_i}{M_{\rm tot}}$$

do i=1,n

pick: y_i

image plane point

calculate: u_i

source plane point

store (u_i <--> y_i)

enddo

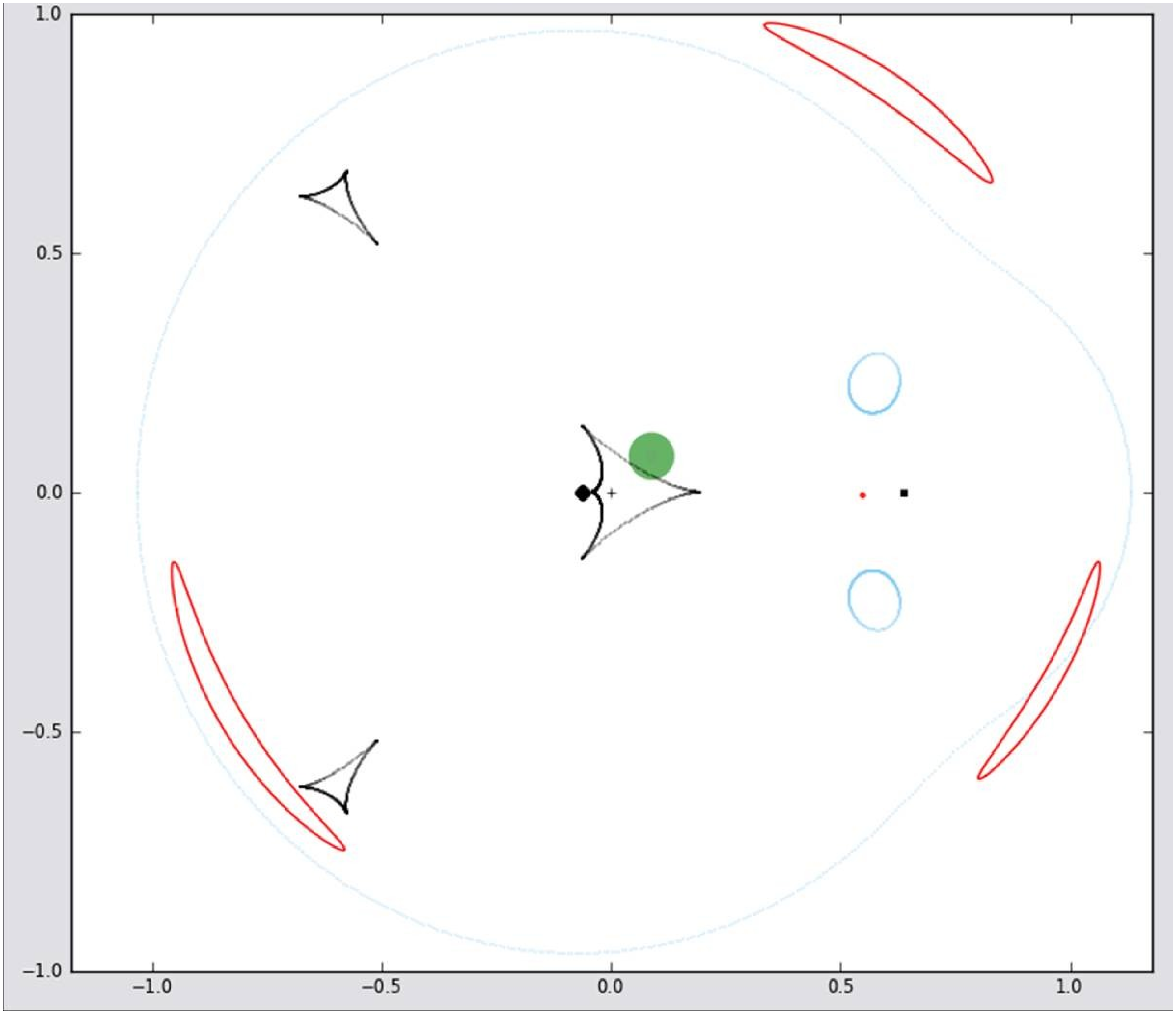
Pick source boundary

Examine each pt u_i

in: weight by LD

out: discard

Sum up weights


1.0
0.5
0.0
−0.5
−1.0
−1.0
−0.5
0.0
0.5
1.0

Inverse Ray Shooting

Advantages

- Automatically includes LD
- Always works

Disadvantage

- Very expensive to shoot lens plane

IRS 1: Map-Making

Dong et al. 2006 ApJ 642 842

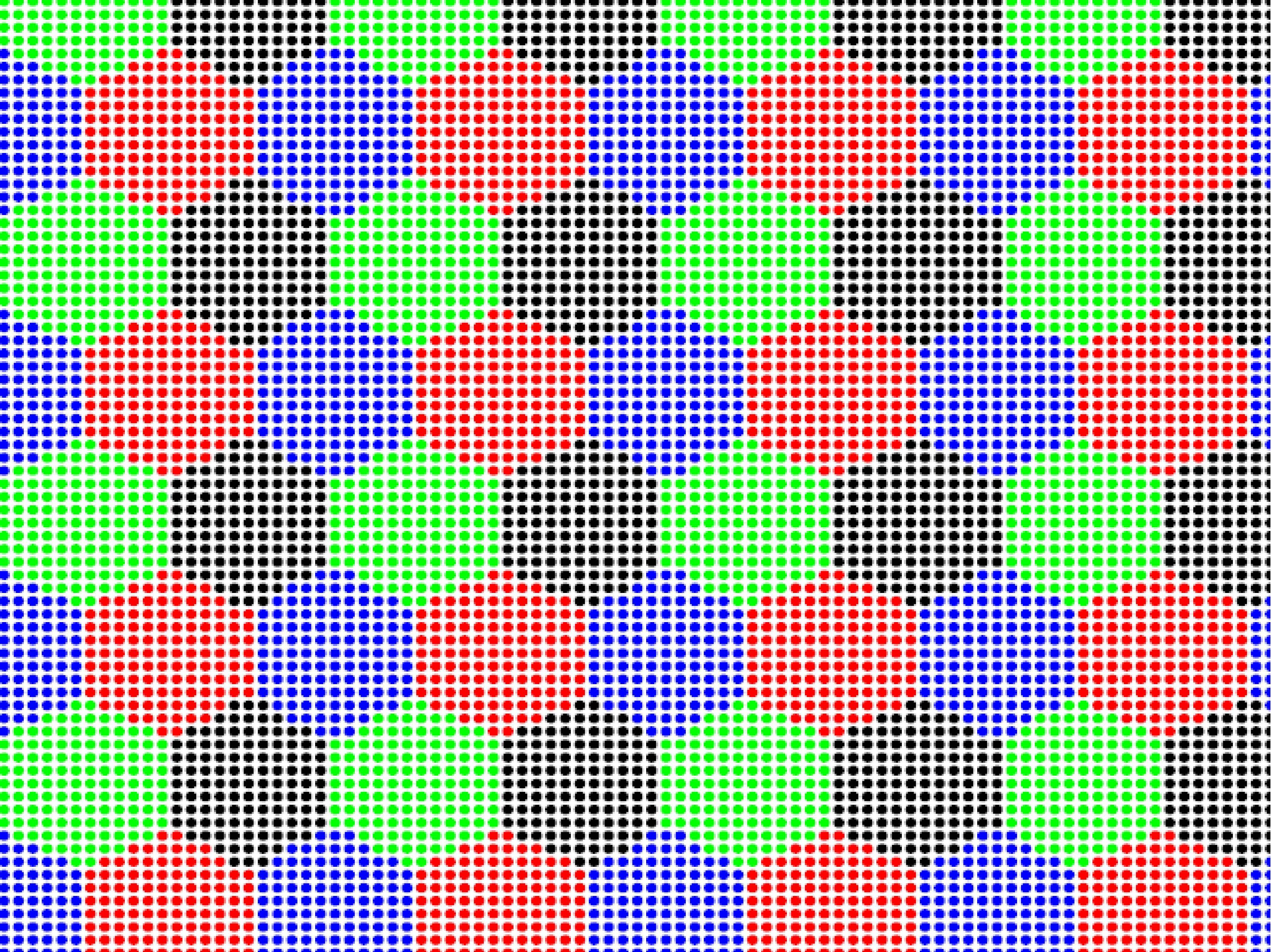
Shoot entire annulus relevant to event

Store rays & hex-tiles on source plane

Use hex-tiles for interior, rays for edge

All (s,q) light curves use ONE map

Disadvantage: requires fixed “s”



IRS 2A: Loop Linking

Dong et al. 2006 ApJ 642 842

Make contour (as in Gould+Gaucherel)

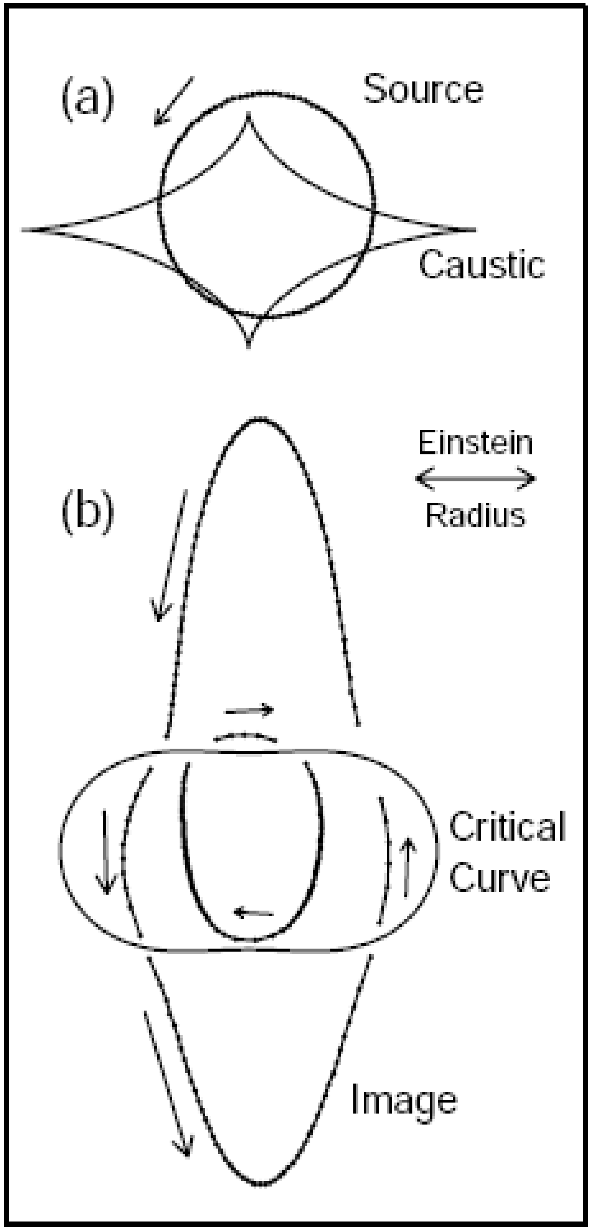
    Except slightly bigger

Shoot rays within images (IRS general)

Advantage: avoids contour problems

Disadvantage: costs more than contour

Contour Integration: Gould & Gaucherel (1997)



$$A = \sum_{i=1}^{n} \sum_{j'} p_j (u_{i-1,j} \times u_{i,j}) \bigg/ \sum_{i=1}^{n} s_{i-1} \times s_i \ ,$$

IRS 2B: Adaptive Images

Bennett 2010, ApJ, 716, 1408

Begin with image centers (point lens)

Expand coverage to source boundary

Radial coord, boundary-sensitive integ.

Advantage: precision with fewer rays

Disadvantage: costs more than contour