Anderson Localization – Looking Forward

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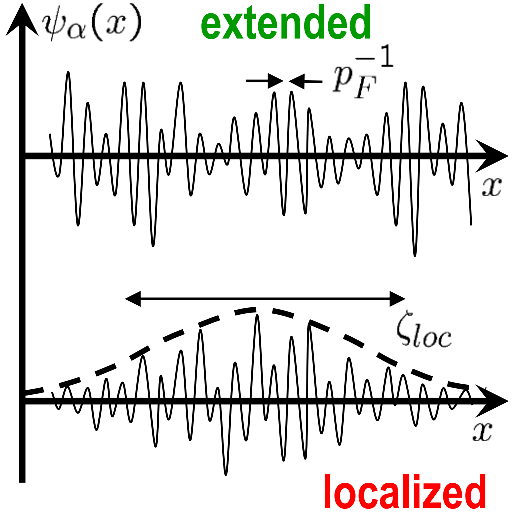
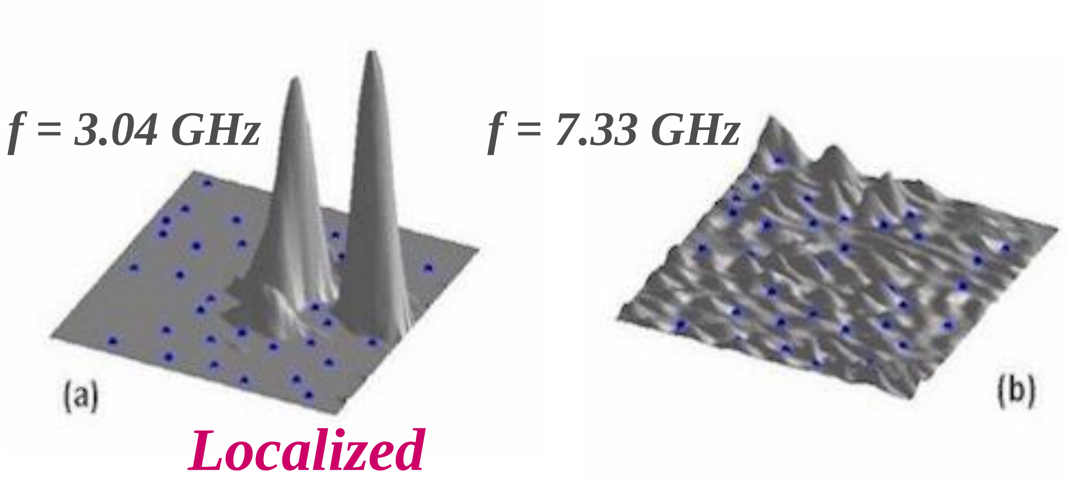
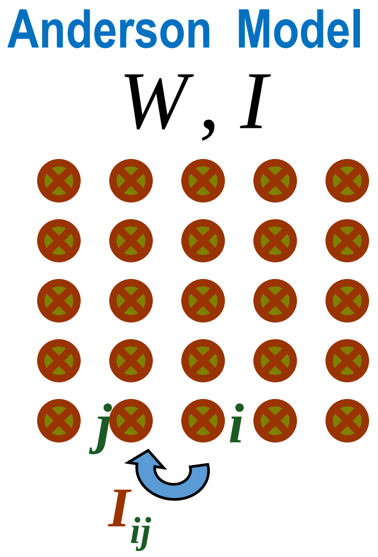
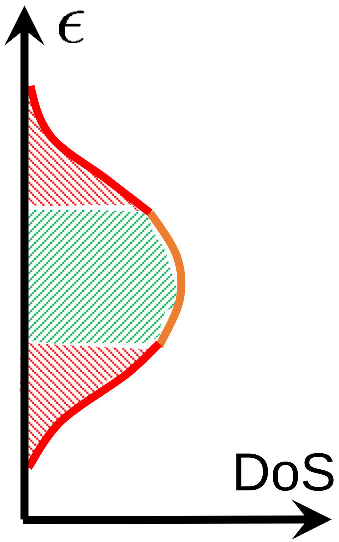
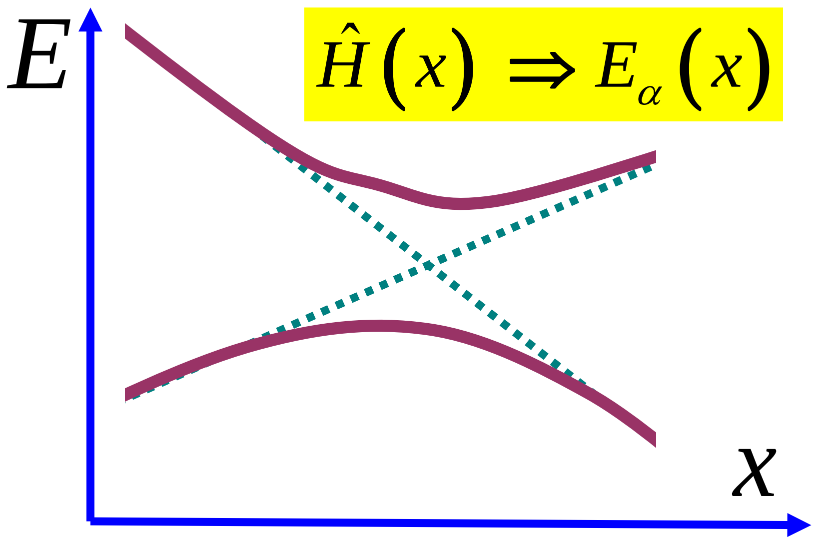
Lecture 4
September, 11, 2015

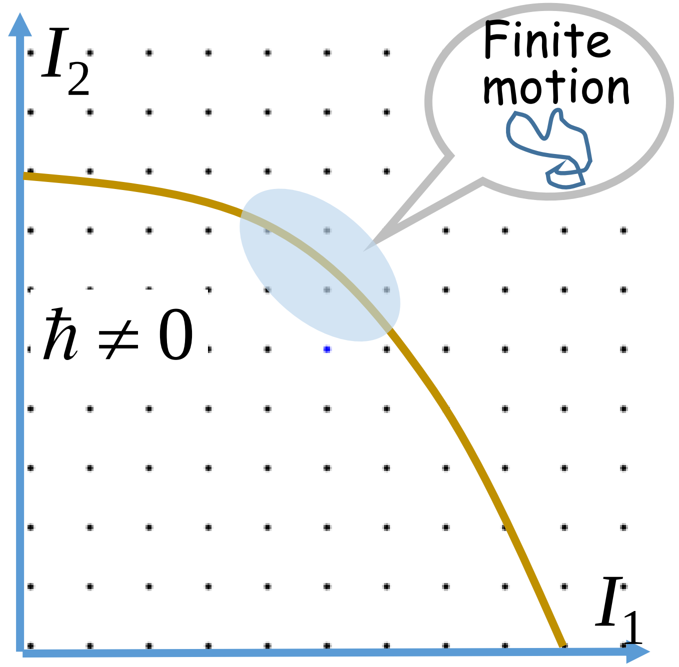

$\psi_\alpha(x)$
extended
p_F^{-1}
x
ζ_{loc}
x
localized

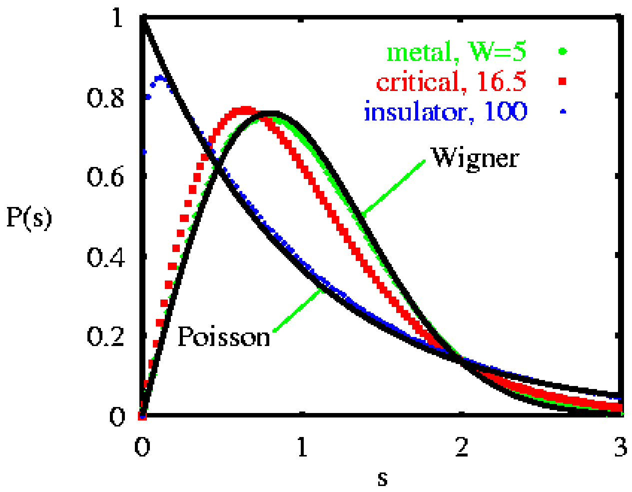

f = 3.04 GHz
f = 7.33 GHz
(a)
(b)
Localized

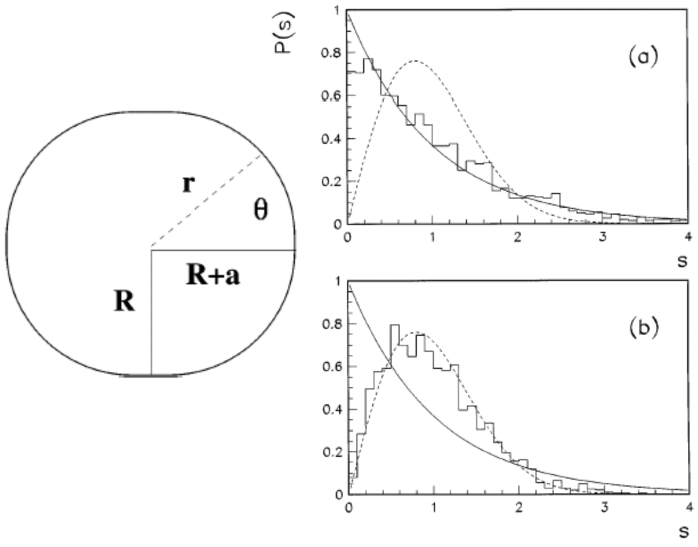

Anderson Model
W, I
j
i
I_{ij}


ϵ
DoS

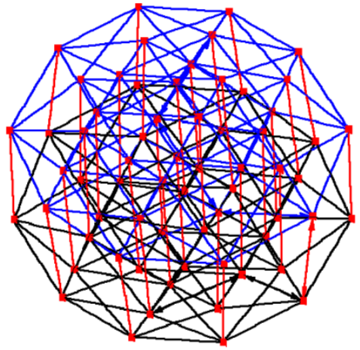

E
$\hat{H}(x) \Rightarrow E_\alpha(x)$
X

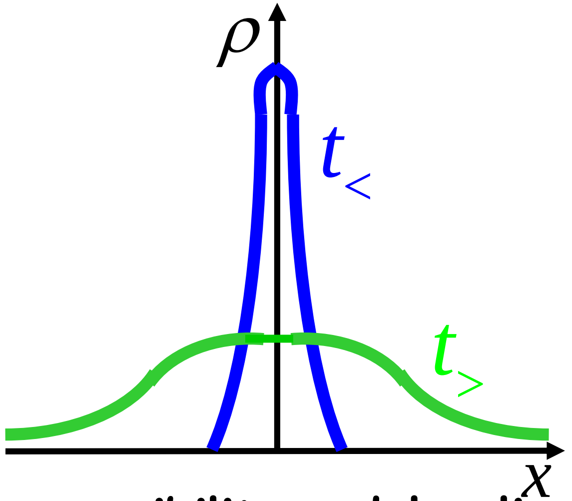

I_2
Finite motion
$\hbar \neq 0$
I_1



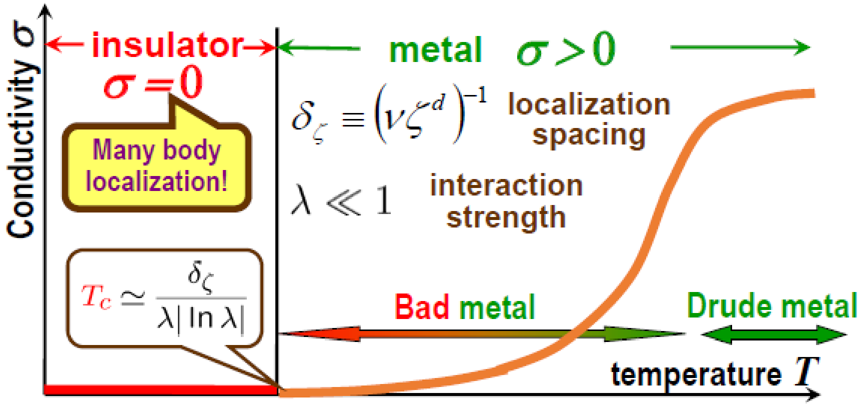


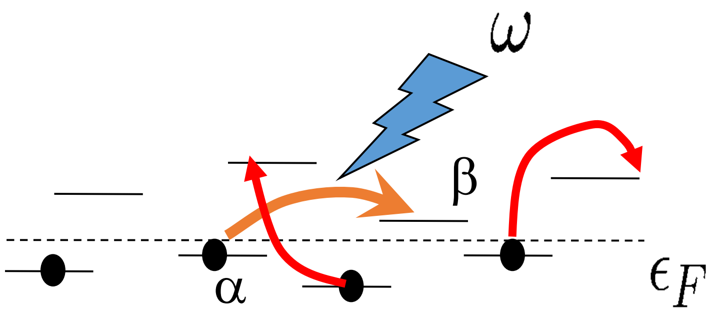
$$\hat{H} = \sum_{i=1}^{N} B_i \hat{\sigma}_i^z + \sum_{i \neq j} J_{ij} \hat{\sigma}_i^z \hat{\sigma}_j^z + I \sum_{i=1}^{N} \hat{\sigma}_i^x \equiv \hat{H}_0 + I \sum_{i=1}^{N} \hat{\sigma}_i^x$$

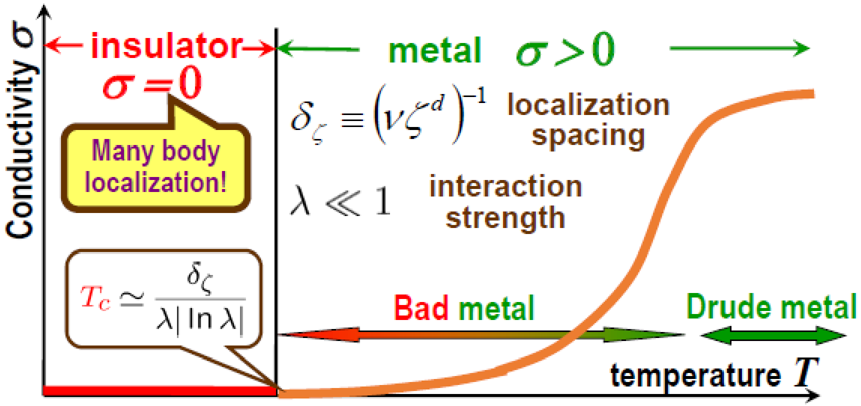


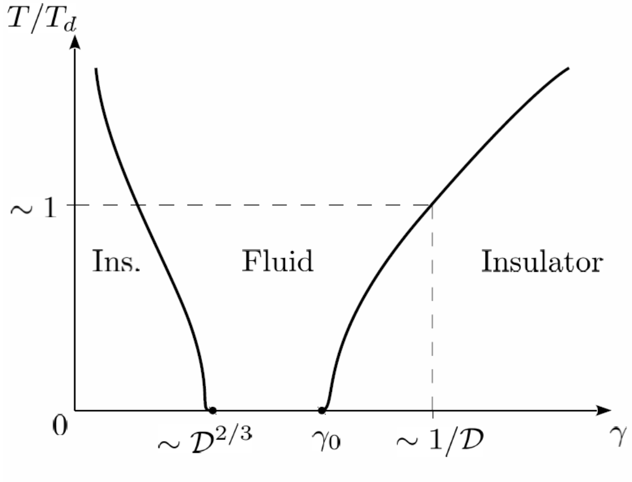


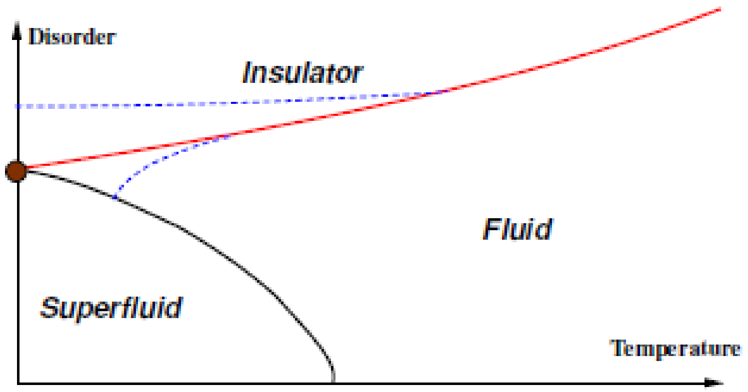
Irreversibility = delocalization

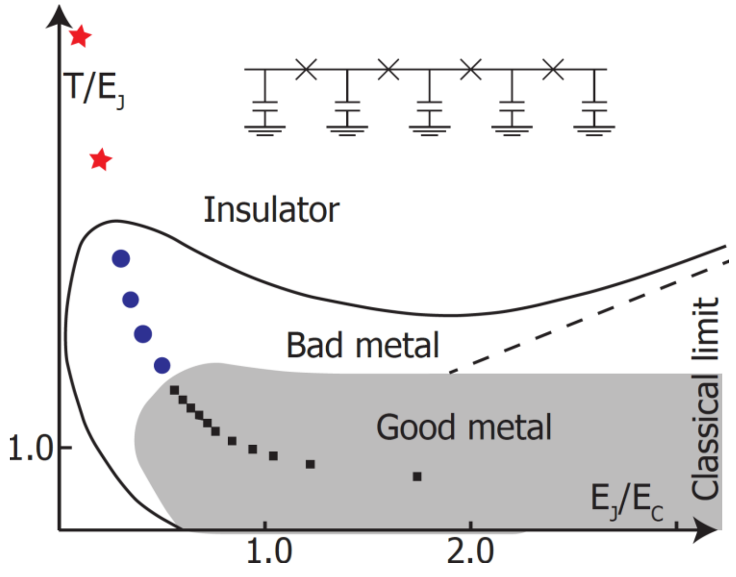


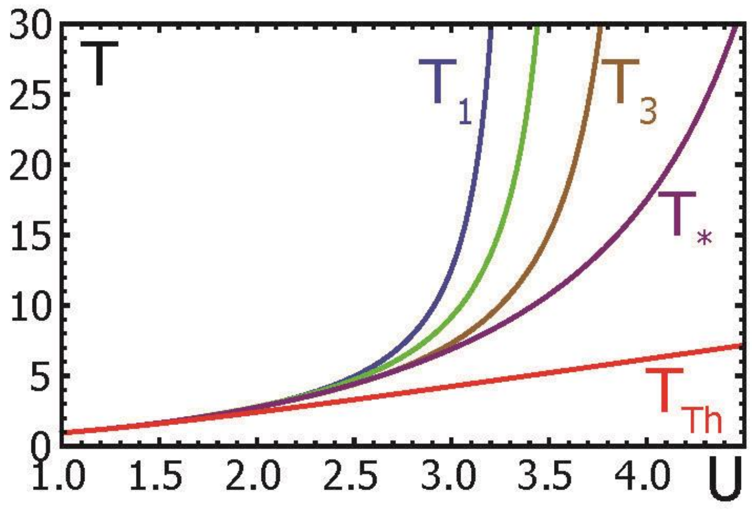


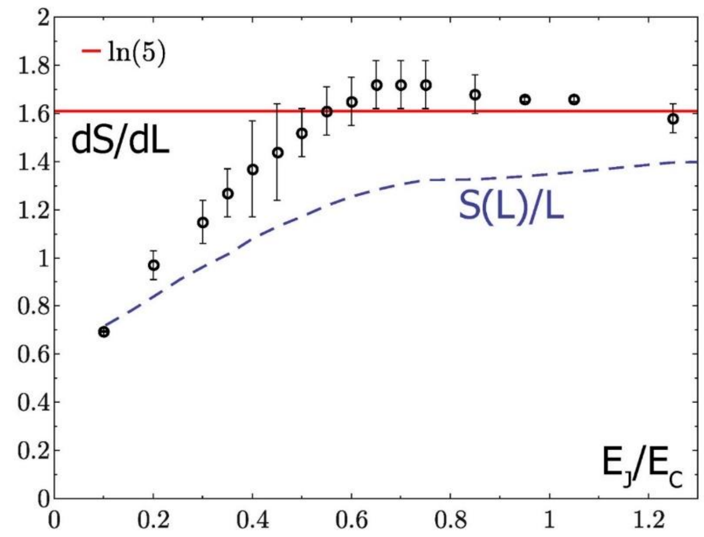

Conductivity σ
insulator
$\sigma = 0$
metal $\sigma > 0$
Many body localization!
$\delta_\zeta \equiv (\nu \zeta^d)^{-1}$
localization spacing
$\lambda \ll 1$
interaction strength
$T_c \simeq \frac{\delta_\zeta}{\lambda |\ln \lambda|}$
Bad metal
Drude metal
temperature T


T/T_d
~ 1
0
Ins.
Fluid
Insulator
$\sim \mathcal{D}^{2/3}$
γ_0
$\sim 1/\mathcal{D}$
γ


Disorder
Insulator
Fluid
Superfluid
Temperature


T/E_J
Insulator
Bad metal
Good metal
Classical limit
1.0
1.0
2.0
E_J/E_C

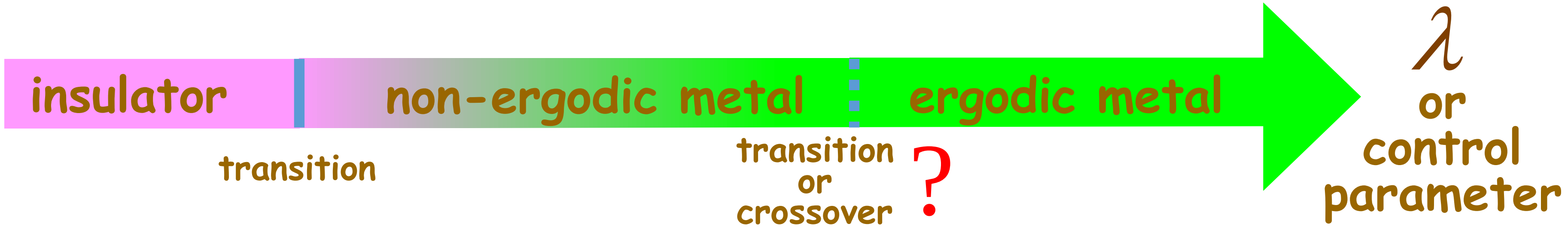

T
T_1
T_3
T_*
T_{Th}
30
25
20
15
10
5
0
1.0
1.5
2.0
2.5
3.0
3.5
4.0
U


ln(5)
dS/dL
S(L)/L
E_J/E_C
2
1.8
1.6
1.4
1.2
1
0.8
0.6
0.4
0.2
0
0.2
0.4
0.6
0.8
1
1.2

Central problem of the proposed research:

Delocalized but not ergodic systems

$$H = H_0 + \lambda V$$



Boltzmann:

Ergodicity means that time-average is equivalent to the space-average

Time-average is equivalent to the energy-shell-average

Quantum: spectrum-average is equivalent to the space-average

Localization and Ergodicity – one particle, $2 < d < \infty$

Anderson Model, N sites $\qquad \hat{H} = \hat{H}_0 + \hat{V}$

$\hat{H}_0 = \sum_i \varepsilon_i |i\rangle\langle i|$ $\qquad \hat{V} = \sum_{i,j=n.n.} |i\rangle\langle j|$ $\qquad -(W/2) < \varepsilon_i < (W/2)$ - random

$\psi_a(i)$ random wave functions

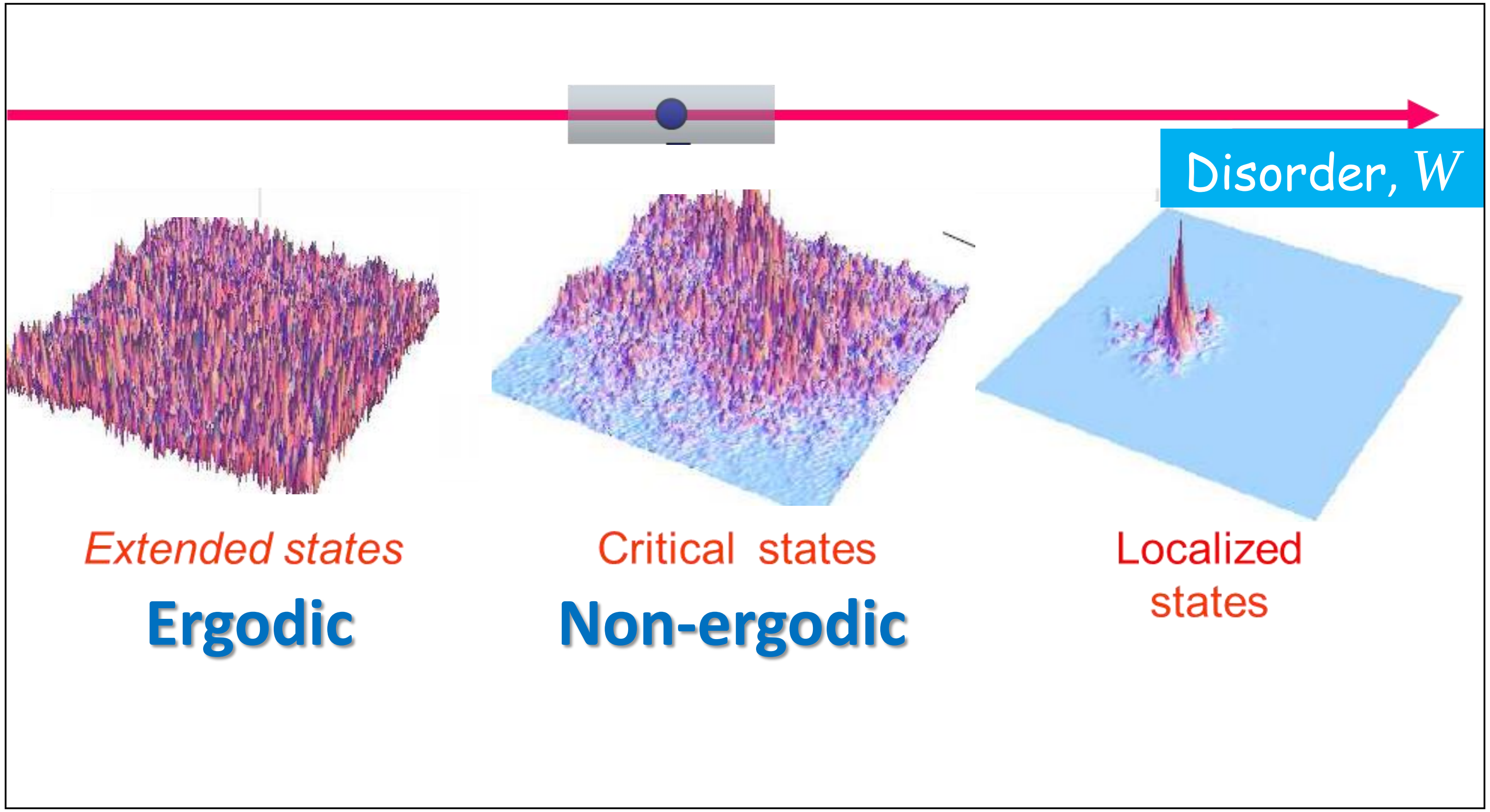
W_c critical disorder

For $N \to \infty$, $W \neq W_c$ fixed, $\psi_a(i)$ are localized if $W > W_c$; extended ergodic if $W < W_c$

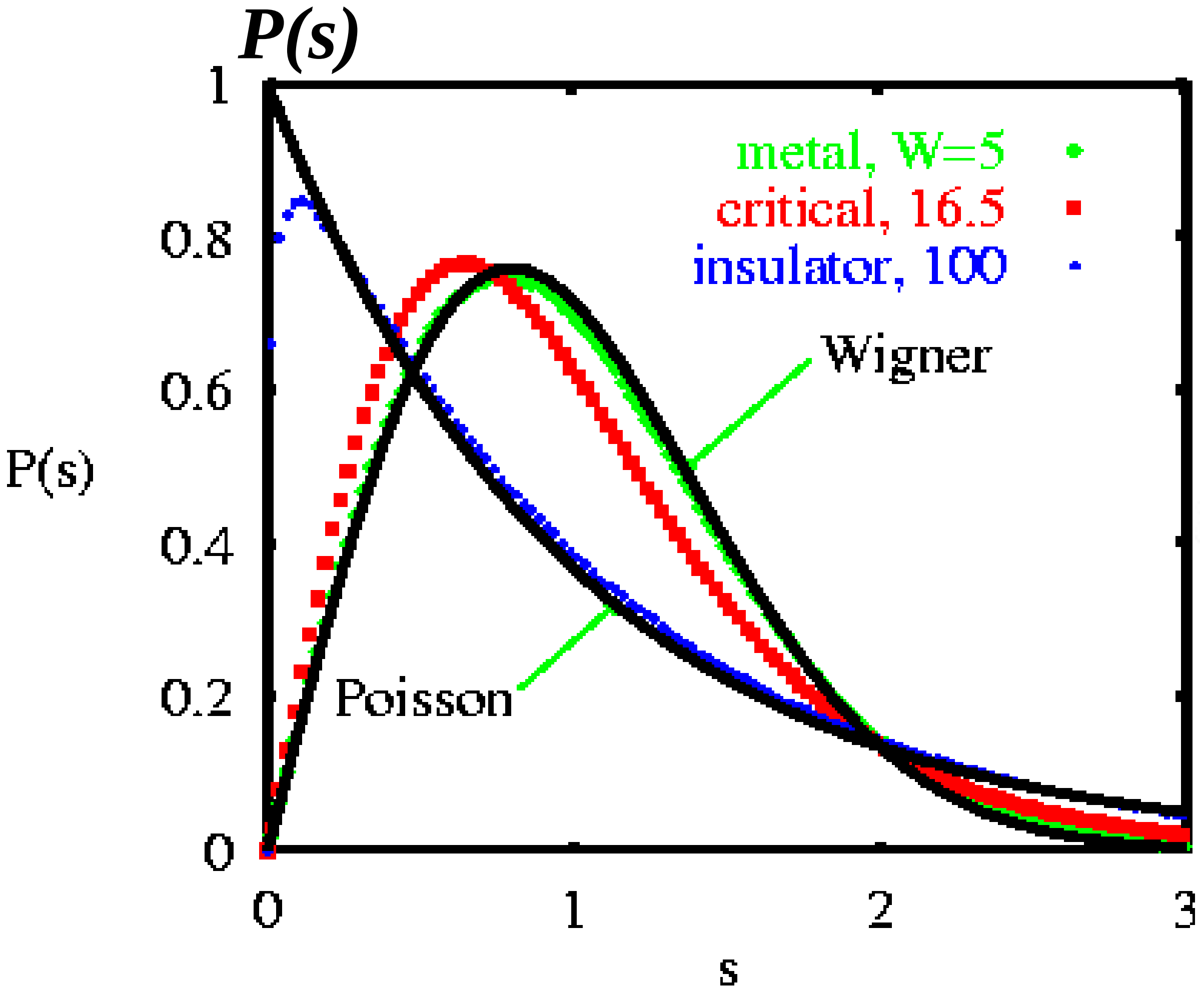
Critical behavior: Critical volume: $N_c(W) \xrightarrow[W \to W_c]{} \infty$

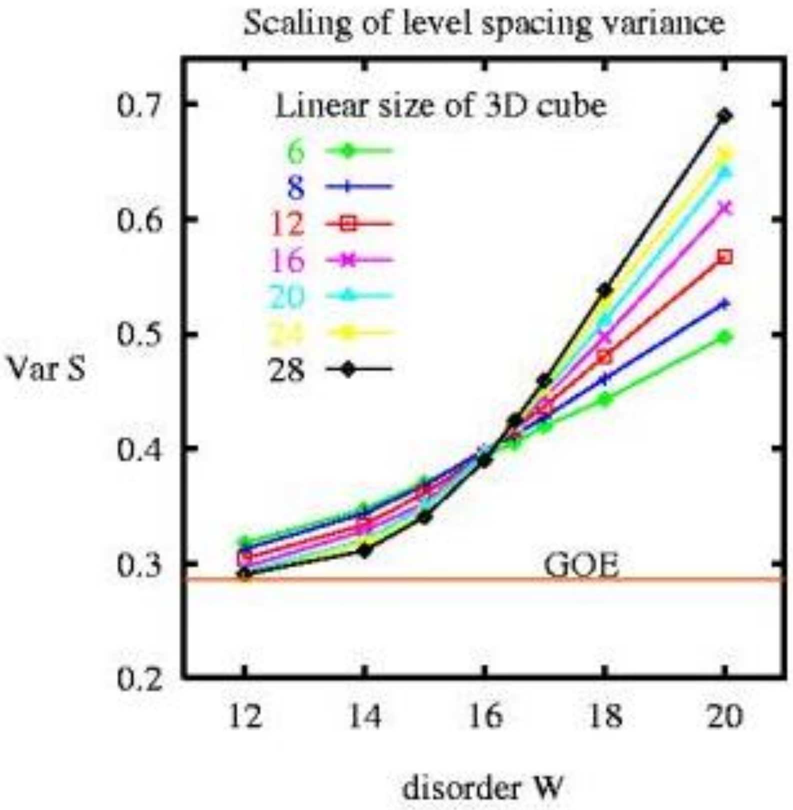
$1 \ll N \ll N_c \Rightarrow \psi_a(i)$ are extended but non-ergodic $\approx$ multifractal

3D Anderson transition



Anderson transition in terms of level statistics 3D





Multifractality

Moments of the inverse participation ratio:

$$I_q(N) \equiv \sum_i |\psi_a(i)|^{2q}$$

$$I_1(N) = 1$$

normalization

Scaling with $N \to \infty$

$$I_q(N) = O(N^0) N^{-\tau(q)}$$

$$\tau_1 = 0$$

Multifractality

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normalization

Scaling with $N \to \infty$

$$I_q(N) = O\left(N^0\right) N^{-\tau(q)}$$

$\tau_1 = 0$

Ergodicity: $\tau(q) = q - 1$ ⟺ $|\psi_a(i)|^2 = O\left(N^{-1}\right)$

Exponentially localized states: $\tau(q) = 0 \quad \forall q$

Multifractality

$$D_q \equiv \frac{\tau(q)}{q-1}$$

Fractal dimensions differ from 0 and 1

They depend on q

Spectrum of fractal dimensions

Distribution function $P(\alpha)$

Statistics of the onsite values of the eigenfunctions $|\psi_a(i)|^2$

$$\alpha_i = -\frac{\ln|\psi_a(i)|^2}{\ln N}$$

random variable

Spectrum of fractal dimensions

Statistics of the onsite values of the eigenfunctions $\left|\psi_a(i)\right|^2$

$$\alpha_i = -\frac{\ln\left|\psi_a(i)\right|^2}{\ln N}$$ random variable

Distribution function $P(\alpha)$

Multifractal ansatz $P(\alpha) \xrightarrow[N\to\infty]{} O(N^0)\times N^{f(\alpha)-1}$

$$f(\alpha) \equiv 1 + \lim_{N\to\infty}\left\{\frac{\ln\left[P(\alpha)\right]}{\ln N}\right\}$$

Spectrum of Fractal Dimensions

Legendre transform of τ_q

Spectrum of fractal dimensions

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Spectrum of Fractal Dimensions

Legendre transform of τ_q

Properties of $f(\alpha)$

- $\int P(\alpha)\,d\alpha = 1$ ⇨ $f_{\max} = 1$
- $|\psi_a(i)|^2 < 1$ ⇨ $f(\alpha)$ is a convex function

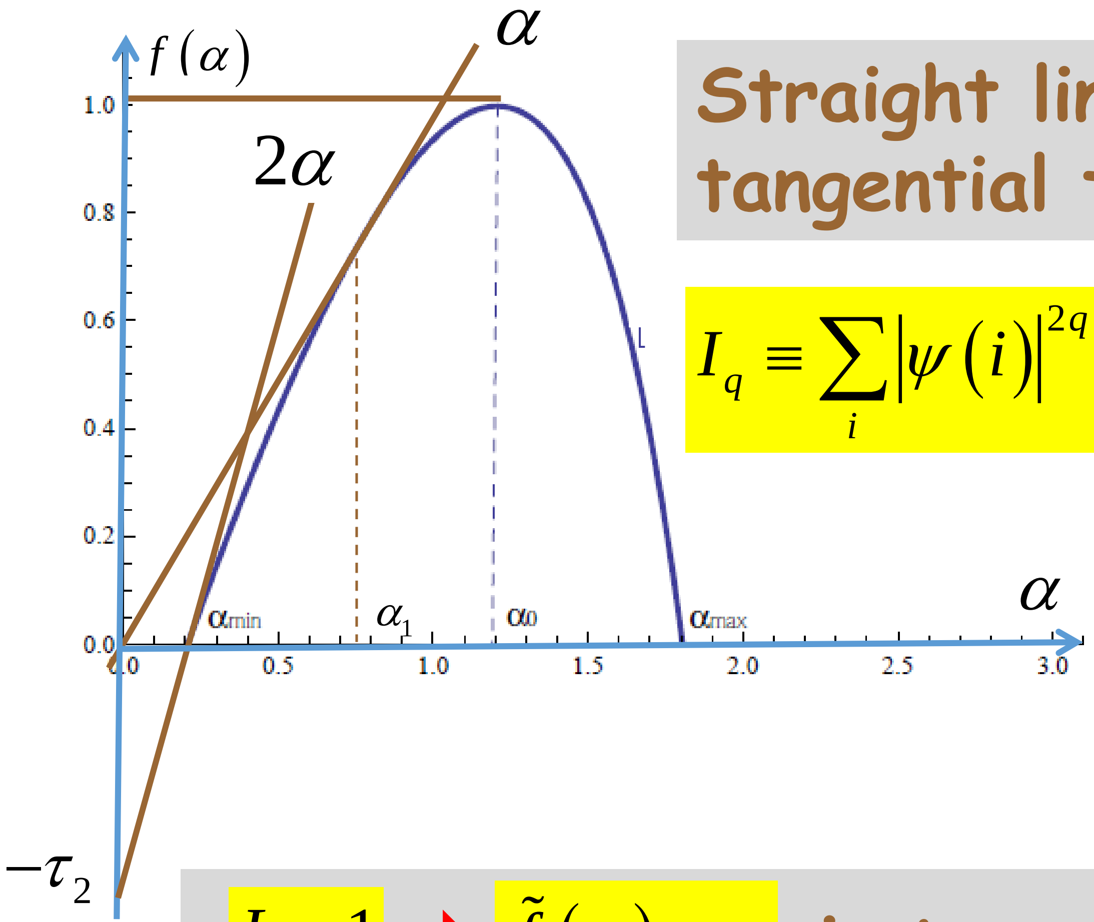
Typical spectrum of fractal dimensions

$$\alpha_q : \left[\frac{\partial f}{\partial \alpha}\right]_{\alpha=\alpha_q} = q$$

$$I_q = N^{-\tau_q}$$

$$\tau_q = q\alpha_q - f(\alpha_q)$$

$$D_q = \frac{q\alpha_q - f(\alpha_q)}{q-1}$$



Straight lines $q\alpha - \tau_q$ are tangential to $f(\alpha)$ at points α_q

$$I_q \equiv \sum_i |\psi(i)|^{2q}$$

$I_0 = N$ ⇨ $\alpha_0 = N$ ⇨ $f_{max} = 1$

$|\psi(i)|^2 < 1$ ⇨ $\partial\tau_q / \partial q > 0$ ⇨ $f(\alpha)$ is convex

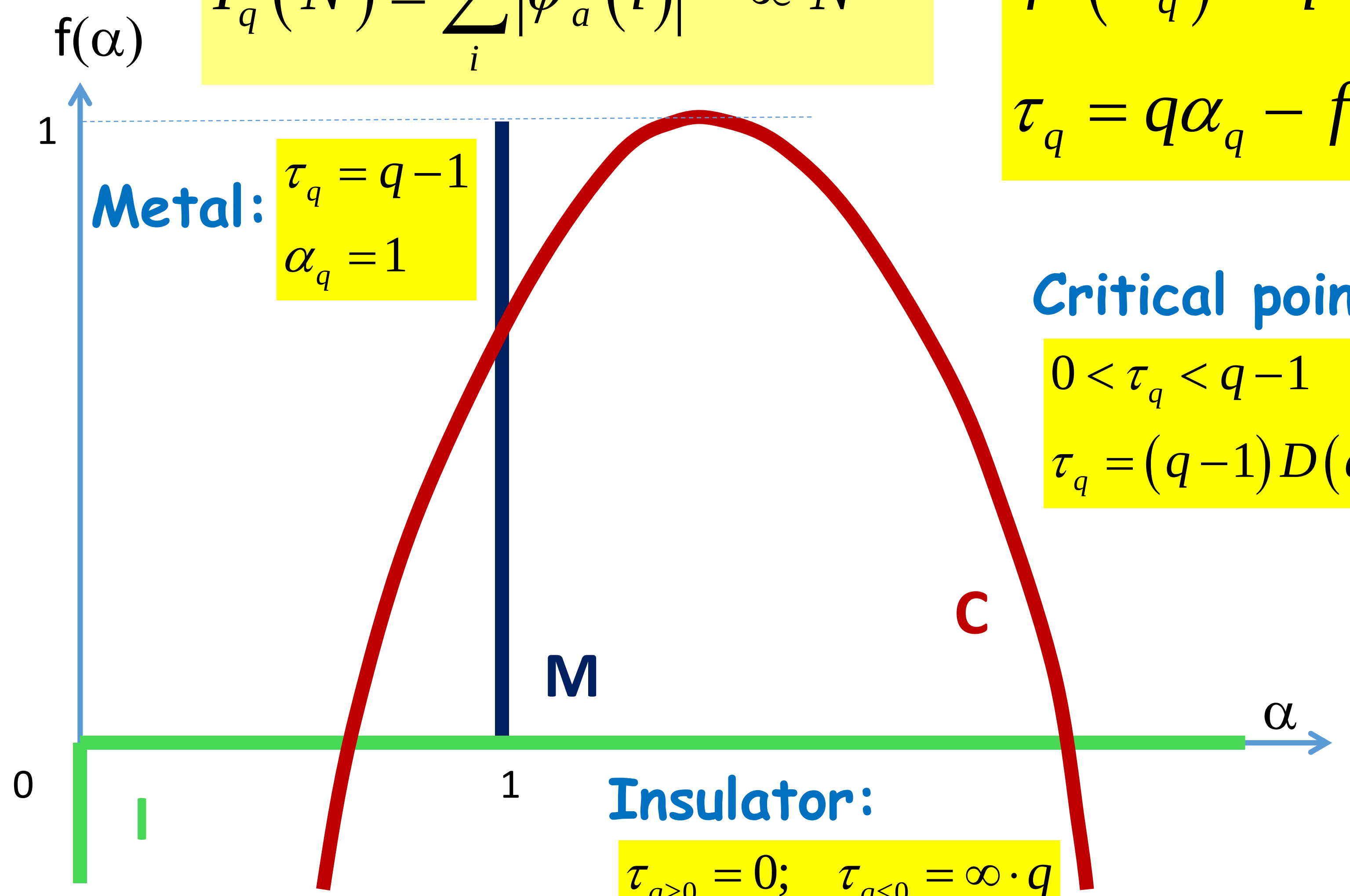
$I_1 = 1$ ⇨ $\tilde{f}(\alpha) = \alpha$ is tangential to $f(\alpha)$ at $\alpha = \alpha_1$

$f(\alpha)$ for a d-dimensional lattice

$$I_q(N) \equiv \sum_i |\psi_a(i)|^{2q} \propto N^{-\tau_q}$$

$$f'(\alpha_q) = q \quad \text{def}$$

$$\tau_q = q\alpha_q - f(\alpha_q)$$



Metal:

$$\tau_q = q - 1$$
$$\alpha_q = 1$$

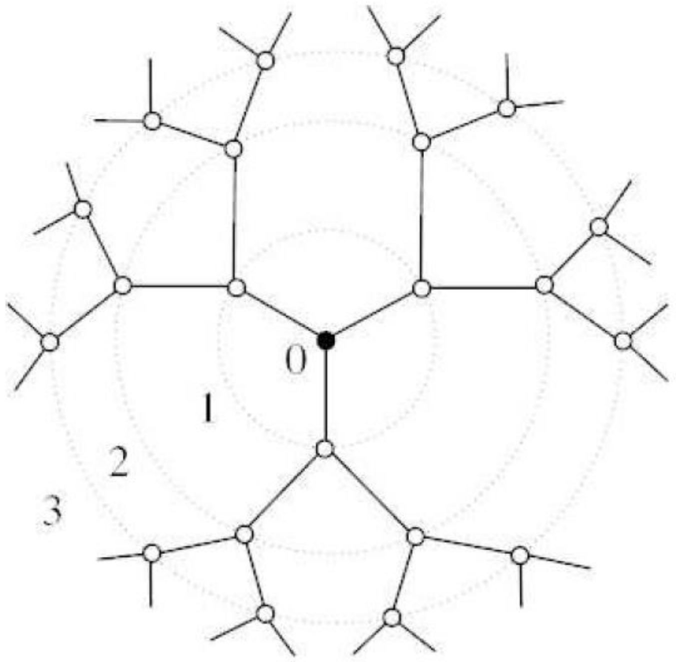
Critical point:

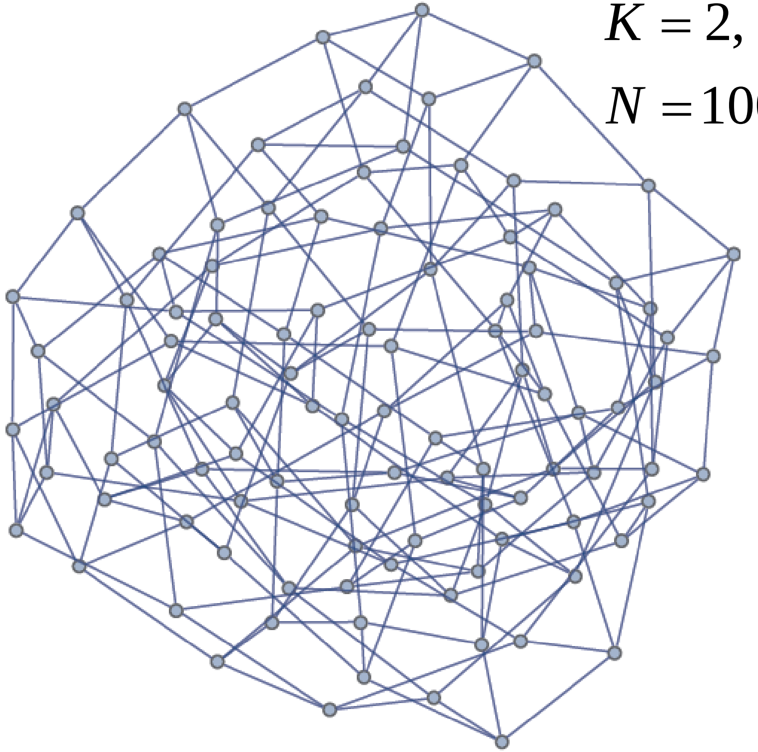
$$0 < \tau_q < q - 1$$
$$\tau_q = (q-1)D(q)$$

Insulator:

$$\tau_{q\geq 0} = 0; \quad \tau_{q\leq 0} = \infty \cdot q$$
$$\alpha_{q\geq 0} = 0; \quad \alpha_{q\leq 0} = \infty$$

Bethe Lattice



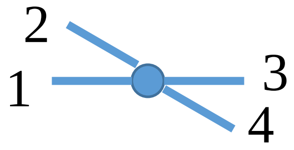


Cayley tree

not good for numeric: most of the sites are on the boundary

Random Regular Graph

with a fixed connectivity **K+1**

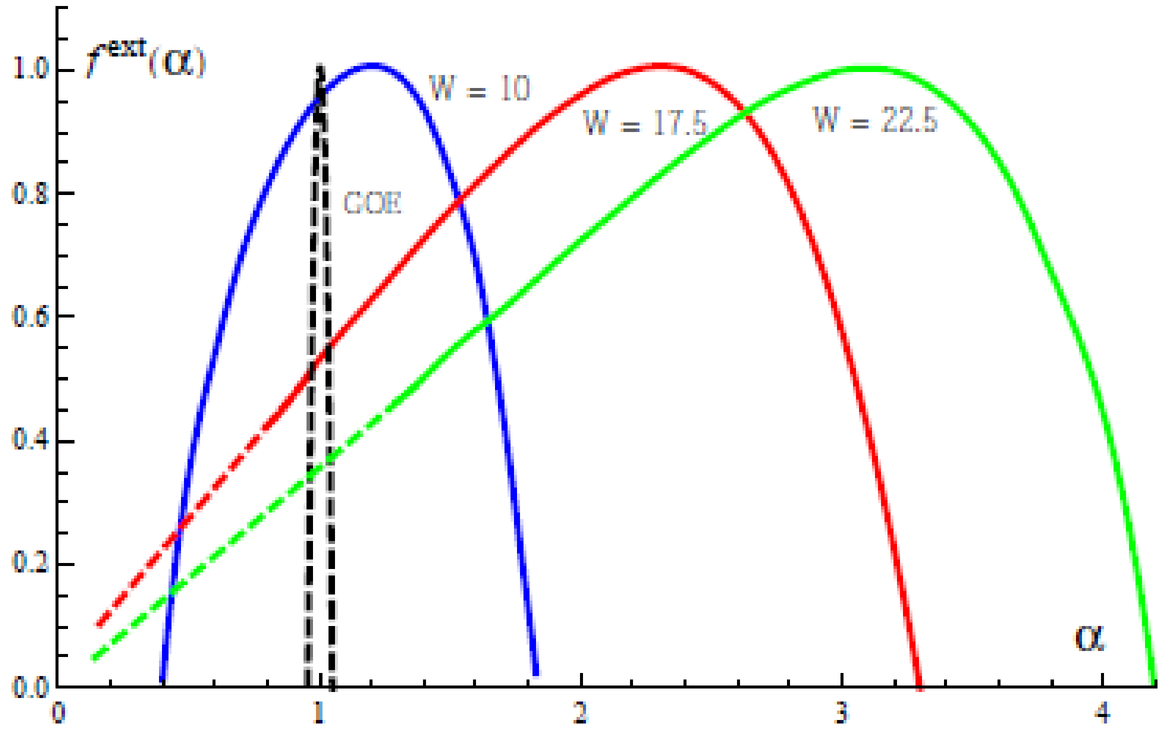


N sites K=3

randomly connected

Q: Can extended eigenstates of the Anderson model on the Bethe-Lattice be non-ergodic outside the critical region **?**

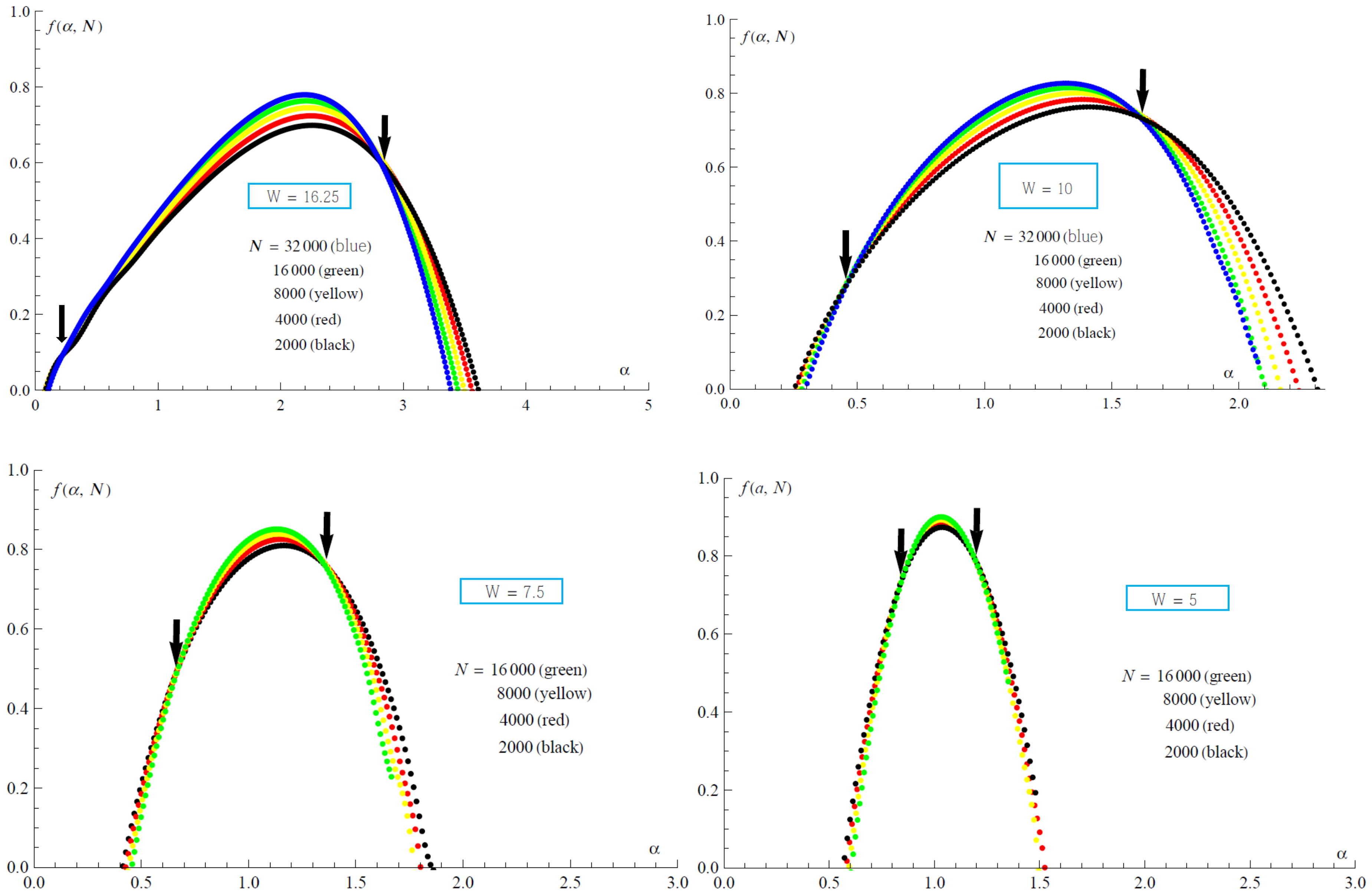
A: YES



Localized states – triangular shape of $f(\alpha)$

Extended states – gradually approach the ergodic limit, but reach it only at $W = 0$

Extended – non-ergodic regime, *W*<17,5:



Extended – non-ergodic regime, $W<W_c=17,5$:

The spectrum of the fractal dimensions $f(\alpha)$ is **gradually** evolving with the strength of disorder W, but does not collapse to the ergodic limit, which is

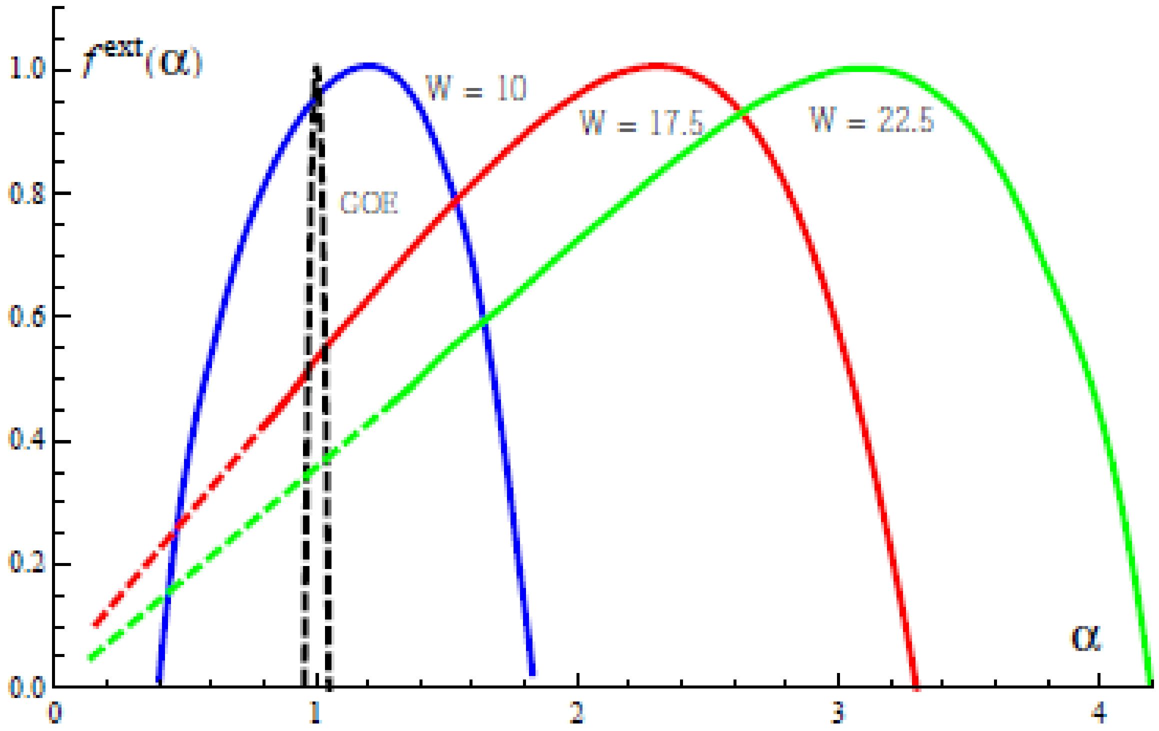
$$f(1)=1 \qquad\qquad f(\alpha\neq 1)=-\infty$$

It is unlikely that this is a finite size effect:

1) Two fixed points
2) This is not a critical behavior: $f(\alpha, N, W)$ **depends on both** N and W.

Q: Can extended eigenstates of the Anderson model on the Bethe-Lattice be non-ergodic outside the critical region ?

A: YES



Localized states – triangular shape of $f(\alpha)$

Extended states – gradually approach the ergodic limit.

When do they reach it reach it Only at $W=0$?

Localization at the Edge of 2D Topological Insulator by Kondo Impurities

2D Topological Insulator

Kane and Mele (2005);

Bernevig, T. L. Hughes, and S. C. Zhang (2006) CdTe-HgTe-CdTe

In the balk (inside the plane) – gap in the spectrum of charge excitations ⇒ insulator

At the edge excitations are gapless ⇒ 1D metal

Insensitive to any static disorder – topological protection.

2D Topological Insulator

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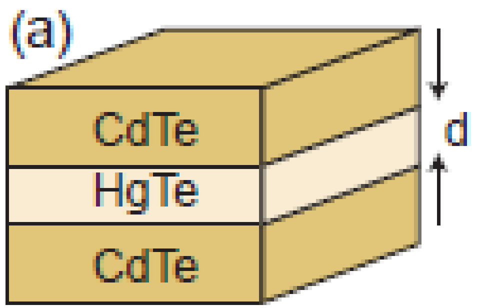
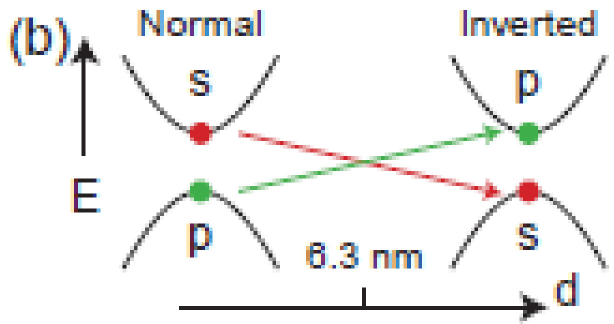
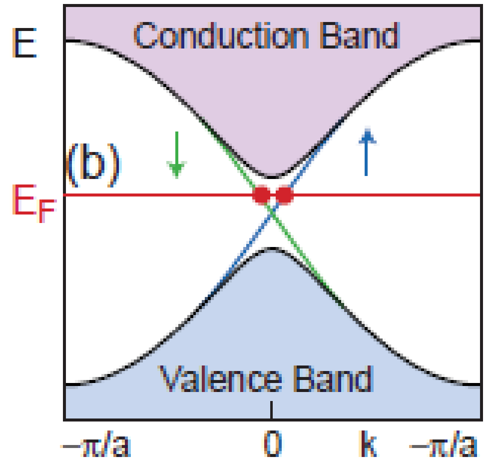






Hamiltonian of the chiral states at the helical edge

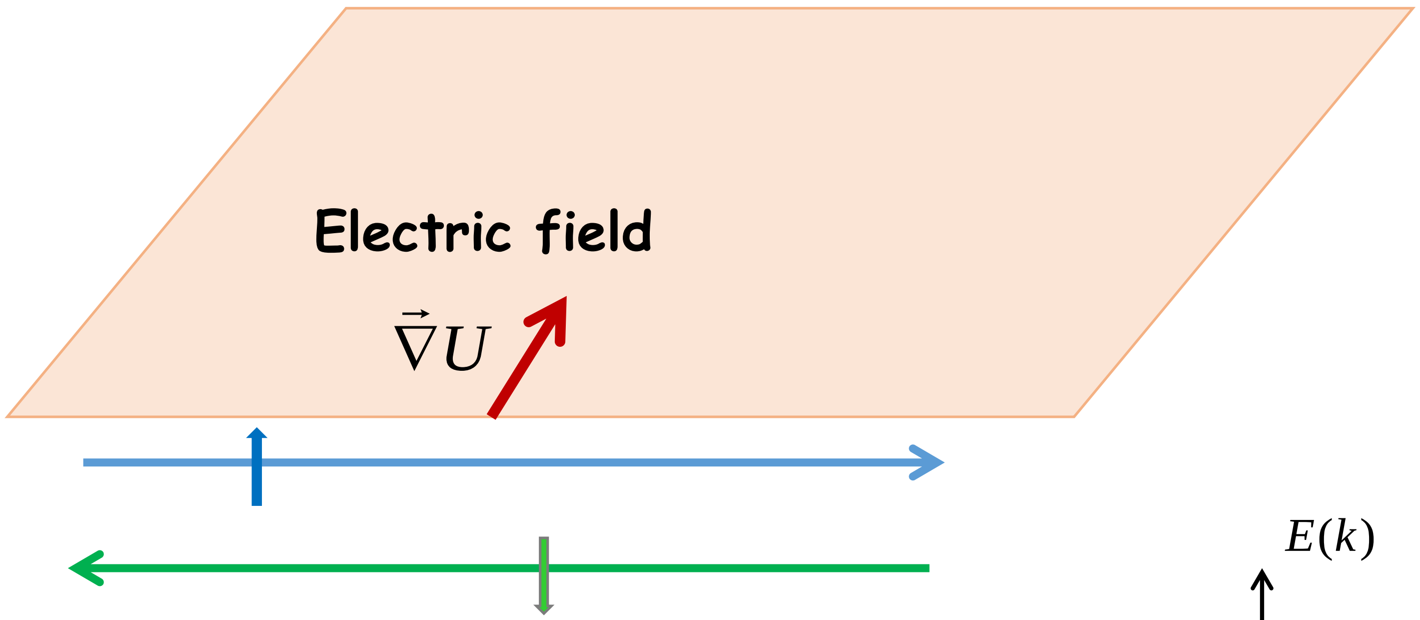
$$\hat{H}(k_x) = v_F k_x \bullet \sigma_z$$

momentum

Fermi velocity

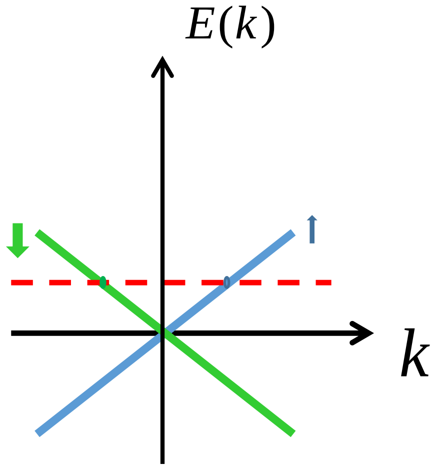
Z-component of the spin

Note: need strong spin-orbital interaction



Rashba term:

$$H_{S-O} \propto \vec{\nabla} U \bullet [\vec{\sigma} \times \vec{k}] \rightarrow \pm v_F k$$



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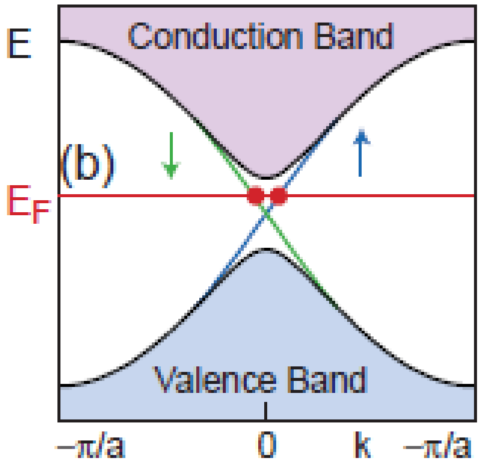
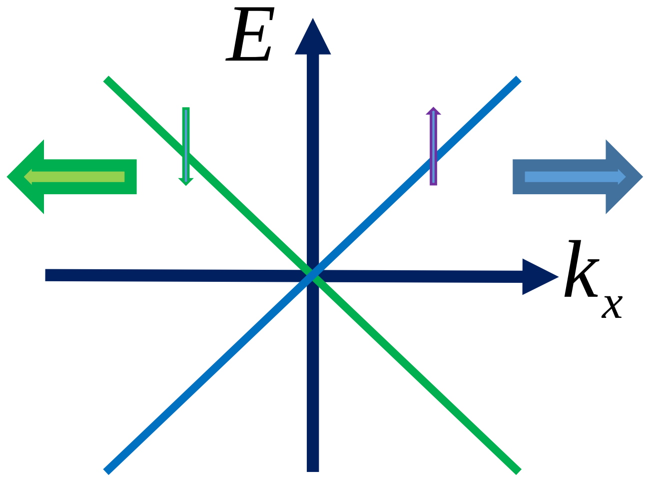


Hamiltonian of the chiral edge states

$$\hat{H}(k_x) = v_F k_x \bullet \sigma_z$$

- k_x: momentum
- v_F: Fermi velocity
- σ_z: Z-component of the spin

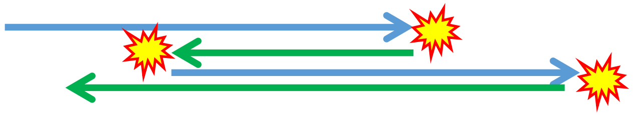


Chiral edge states:

Left and Right movers

1d localization

Mott and Twose (1961): **Even a "*weak*" disorder is not weak in d=1 !**



M. E. Gertsenstein and V. B. Vasiliev, "Wave guides with random inhomogeneities and Brownian motion in the Lobachevsky plane", Theory of Probability & Its Applications, **1959**, Vol. 4, No. 4 : pp. 391-398

Exact Solution!

Localization length ~ mean free path

Consequence: a zero DC conductivity

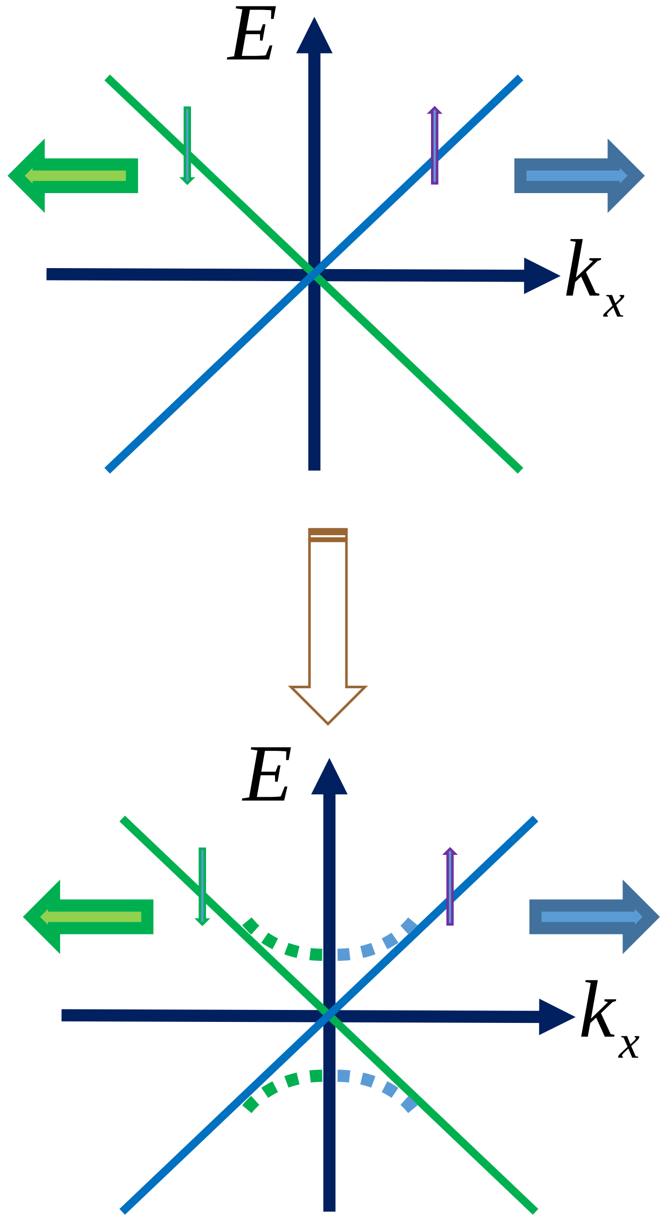
Why all this is not directly applicable to 1D helical edge electrons?

Chiral edge states:
Left and Right movers

Backscattering would mix the chiral states and thus destroy chirality.

One needs spin-flip for the backscattering.

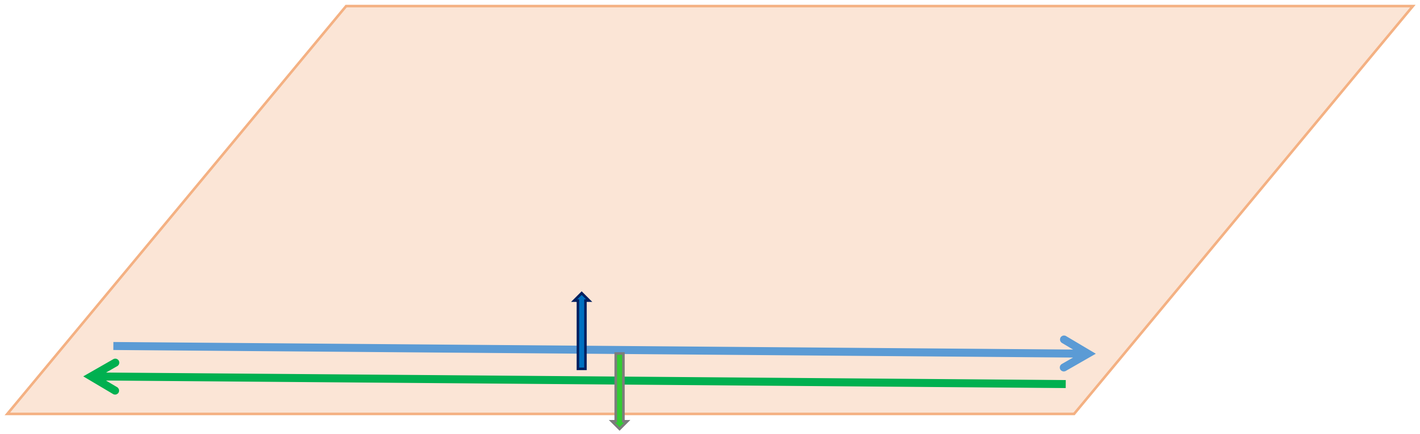
Kramers degeneracy: one needs to violate the time-reversal symmetry to mix left and right movers



Basic properties of a generic 2D Topological Insulator:

(strong spin-orbit coupling)

2D bulk = insulator: electron spectrum is gapped,
levels of impurities are **localized**

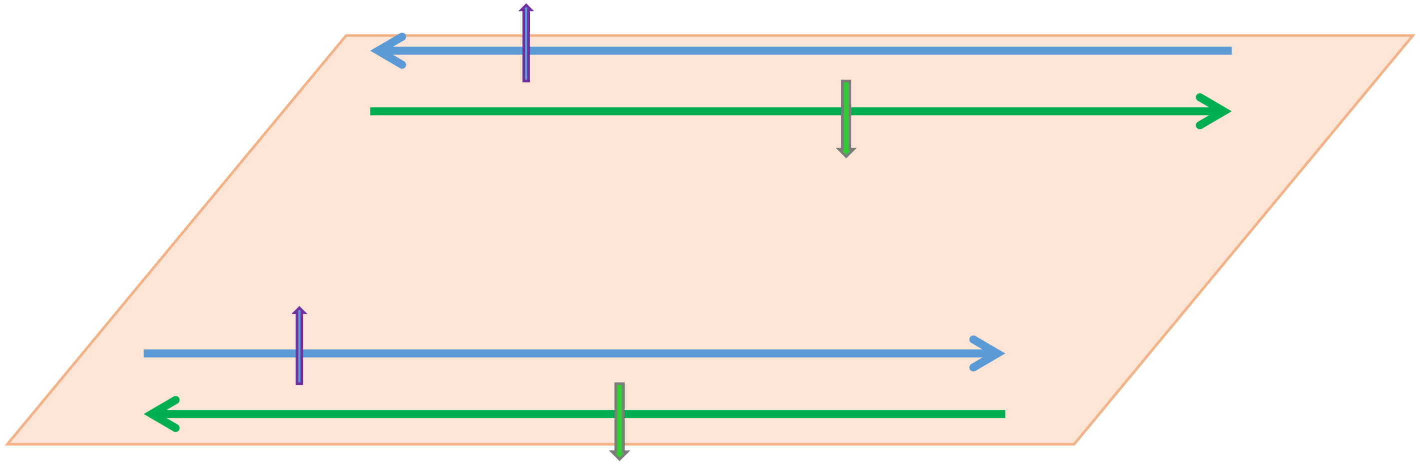


Edge Modes are Helical

Statement:

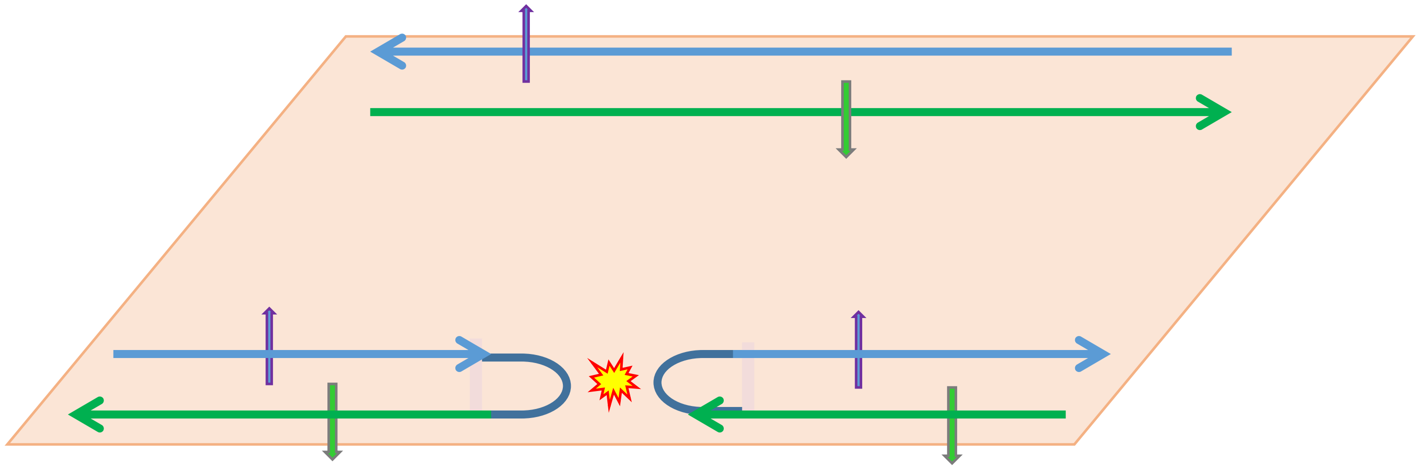
Time Reversal Symmetry protects Helical Edge Modes from Backscattering and thus from localization by a potential disorder

Quantum Hall Effect

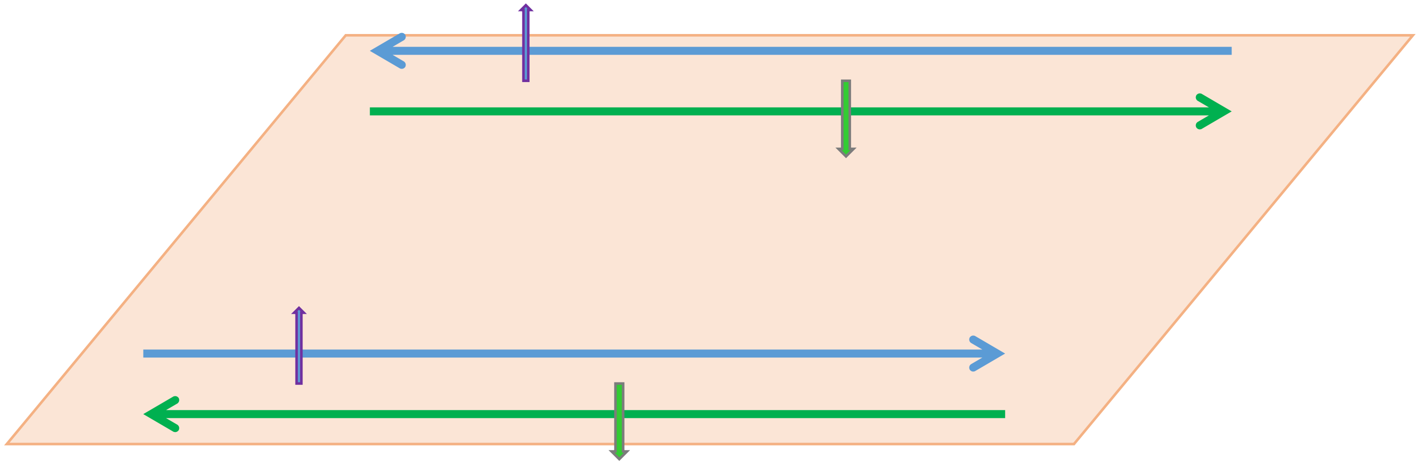


Topological Insulator

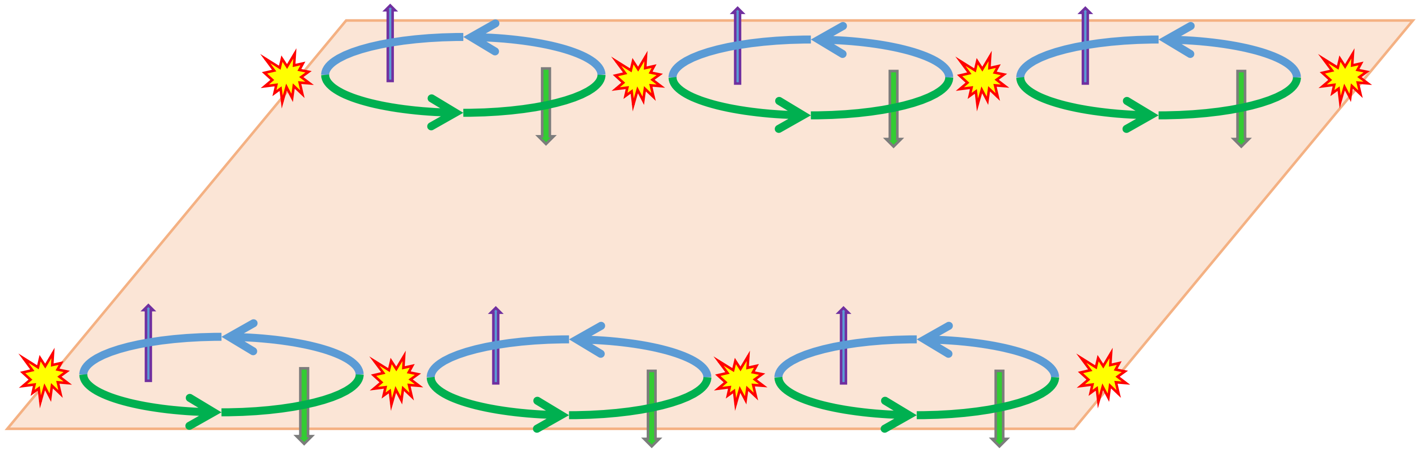
Quantum Hall Effect: spatially separated edge states - nowhere to scatter



Topological Insulator: the back moving state is nearby
but
the back-scattering can not happen without a spin flip



Topological Insulator: metallic edge



Localization at the edge

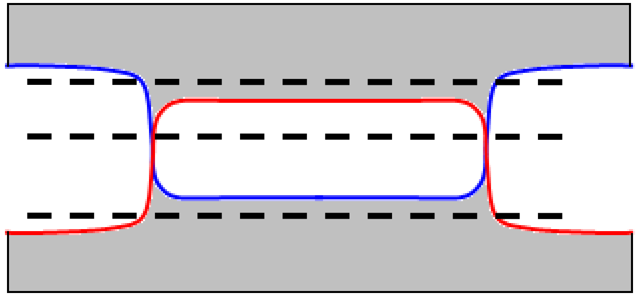
Time Reversal Symmetry protects Helical Edge Modes from Backscattering and Anderson Localization

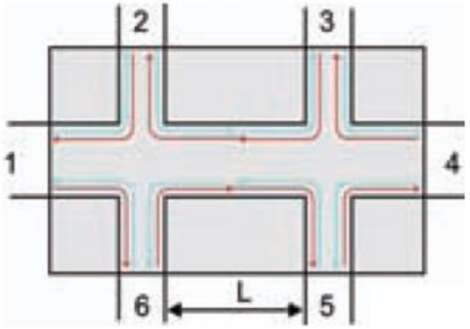
Conductance of an ideal 1D helical edge should be:

$$G_{ideal} = \frac{2e^2}{h}$$

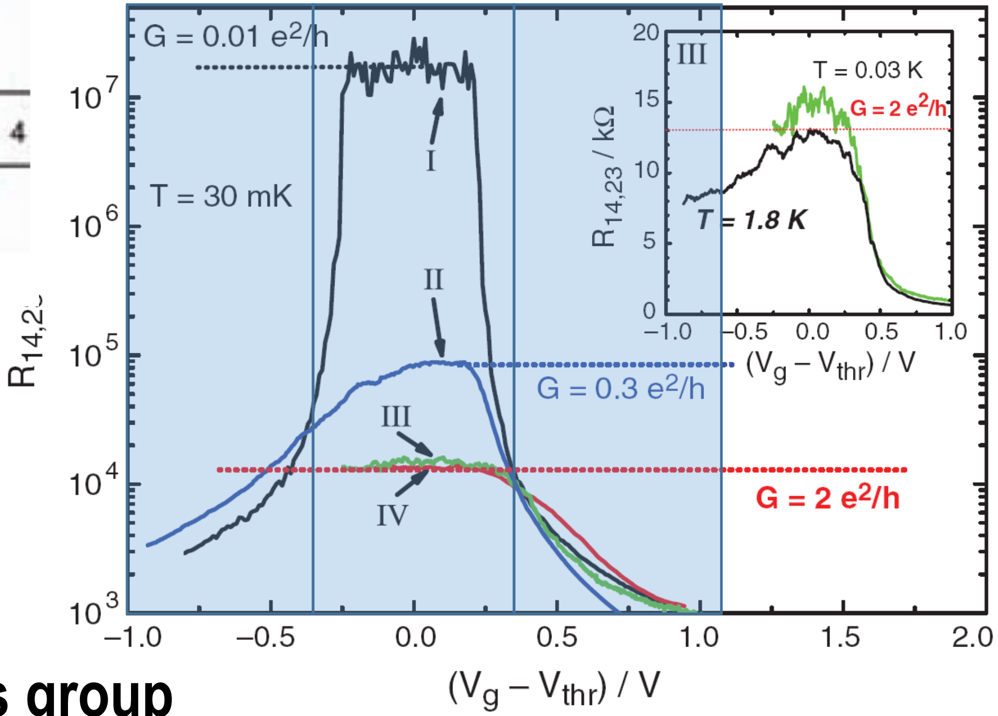
Experimental Observations

- Large samples show large resistance at the gap.
- Small samples (~*1X1μm*) show quantized conductance at the gap, indicating transport by edge states.
- g: 20-50





	d (nm)	L×W (μm²)
I	5.5	20.0×13.3
II	7.3	20.0×13.3
III	7.3	1.0×1.0
IV	7.3	1.0×0.5

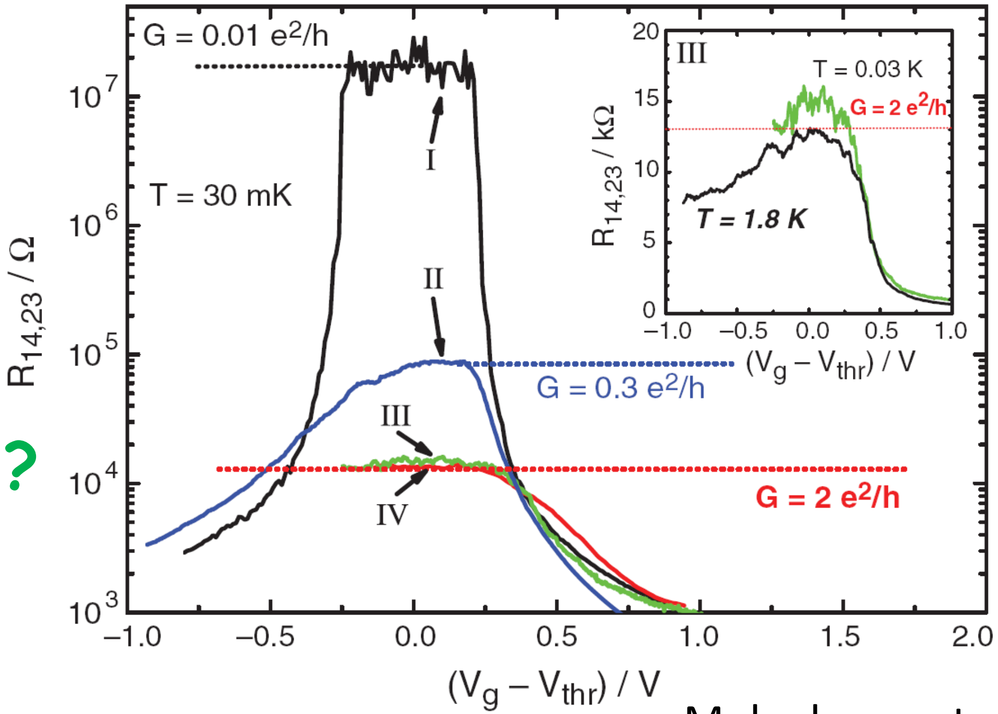


Molenkamp's group

Problems with the interpretation:

- Why the "quantization" of the conductance takes place only in short samples?

	d (nm)	L×W (μm²)	
I	5.5	20.0×13.3	NO
II	7.3	20.0×13.3	NO
III	7.3	1.0×1.0	YES ?
IV	7.3	1.0×0.5	YES



Molenkamp et al.

Time Reversal Symmetry protects Helical Edge Modes from Backscattering and Anderson Localization

Conductance of an ideal 1D helical edge should be:

$$G_{ideal} = \frac{2e^2}{h}$$

In reality, conductance of only **extremely small fraction of short** samples is somewhat close to G_{ideal} , while for most of the short samples and all of the long samples $G << G_{ideal}$

Problems with the interpretation:

- Why the "quantization" of the conductance takes place only in short samples?

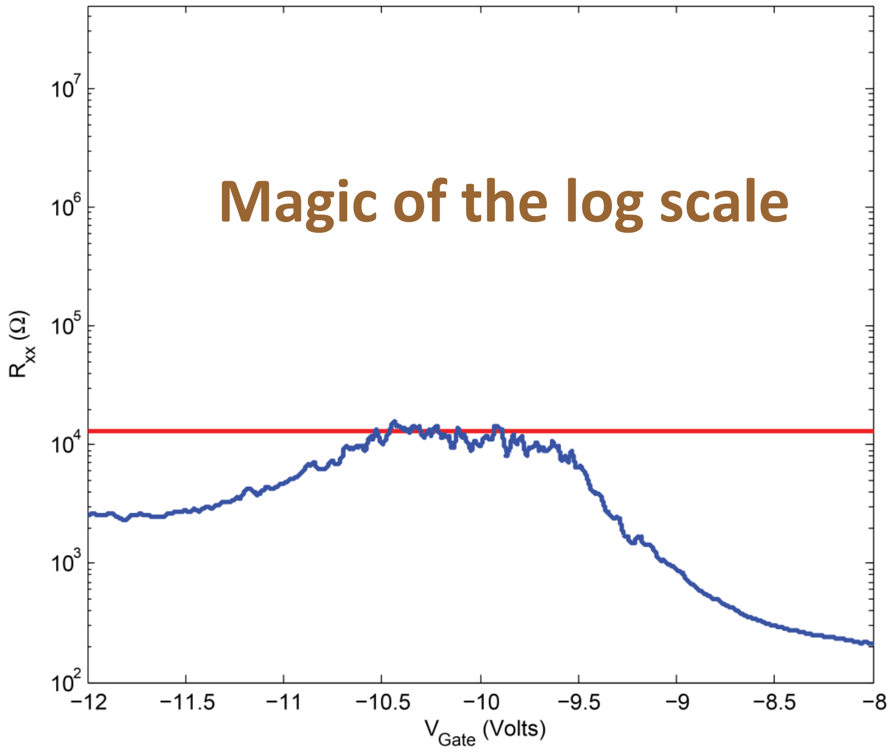
	d (nm)	L×W (μm²)
I	5.5	20.0×13.3
II	7.3	20.0×13.3
III	7.3	1.0×1.0
IV	7.3	1.0×0.5

NO

NO

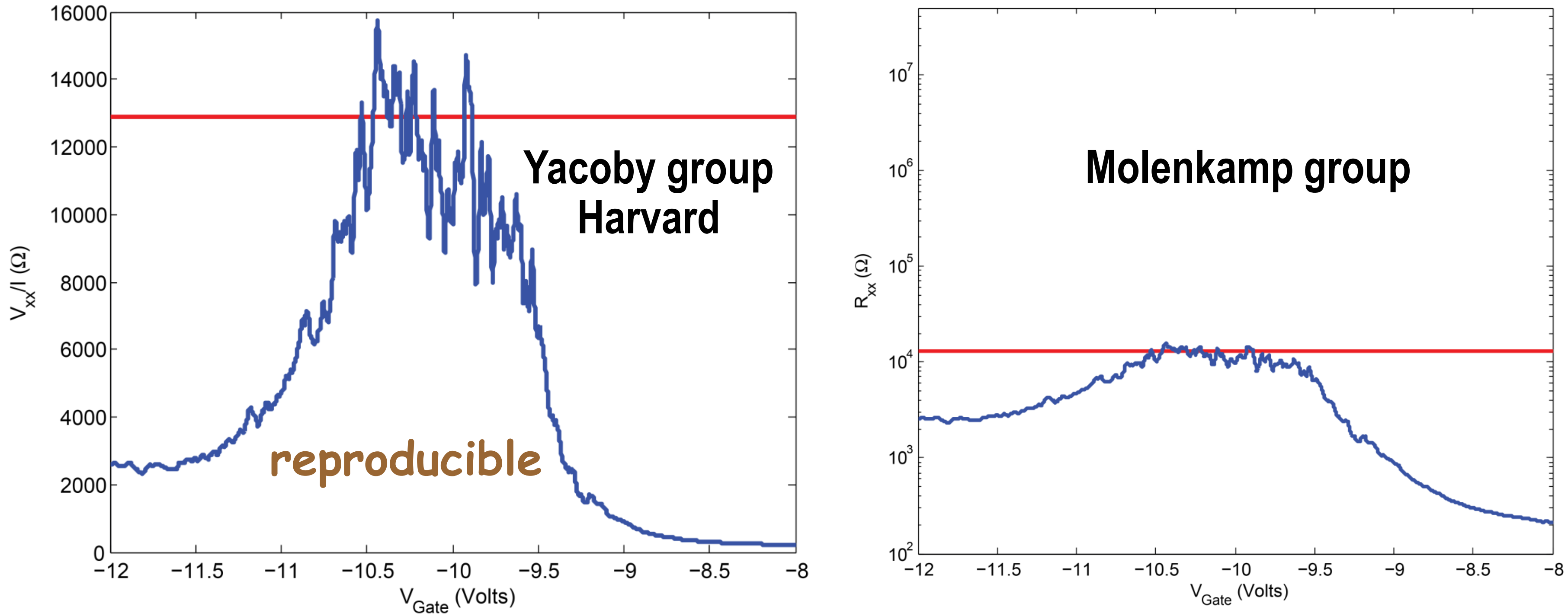
YES

YES



Problems with the interpretation:

- Why the "quantization" of the conductance takes place only in short samples?



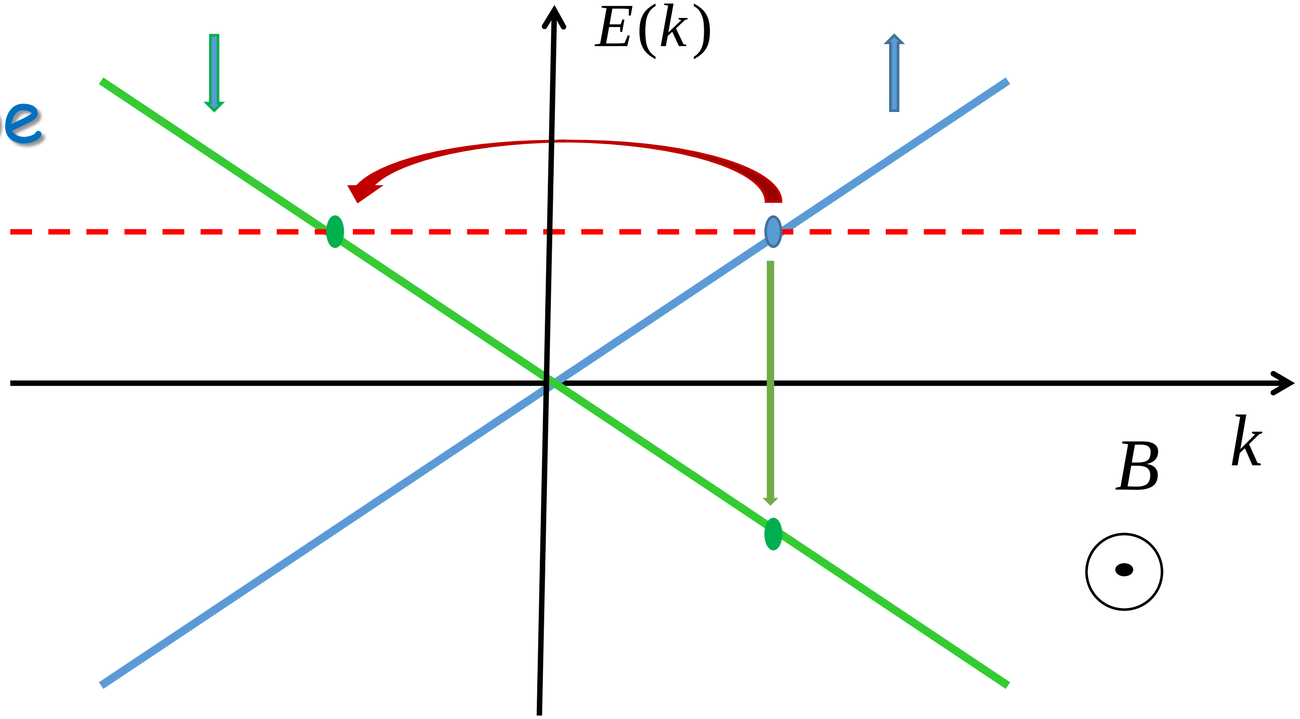
The same sample !

- Why the accuracy of the quantization is so poor? Impressive only in the log scale

Questions:

- How universal is the protection?
- Can the “topological protection“ be softened?
- Can helical edge electrons be localized?
- Role of many-body effects?

Time Reversal Symmetry can be trivially broken by an External Magnetic Field



Spatially homogeneous field – no effect!

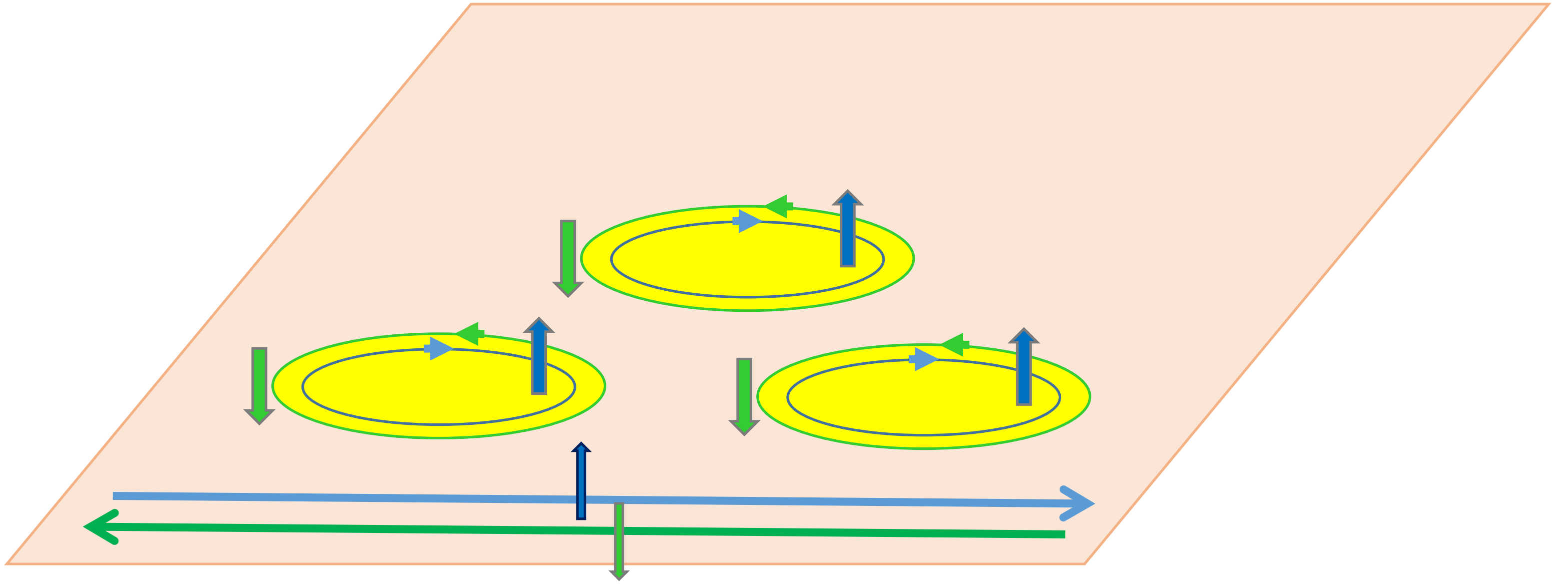
Modulated $\sim \cos(2kx)$ field –> energy gap

Homogeneous field + potential disorder = backscattering!

Q: What about intrinsic sources of the Time Reversal Symmetry Violation ?

Formation of the Kondo Spins

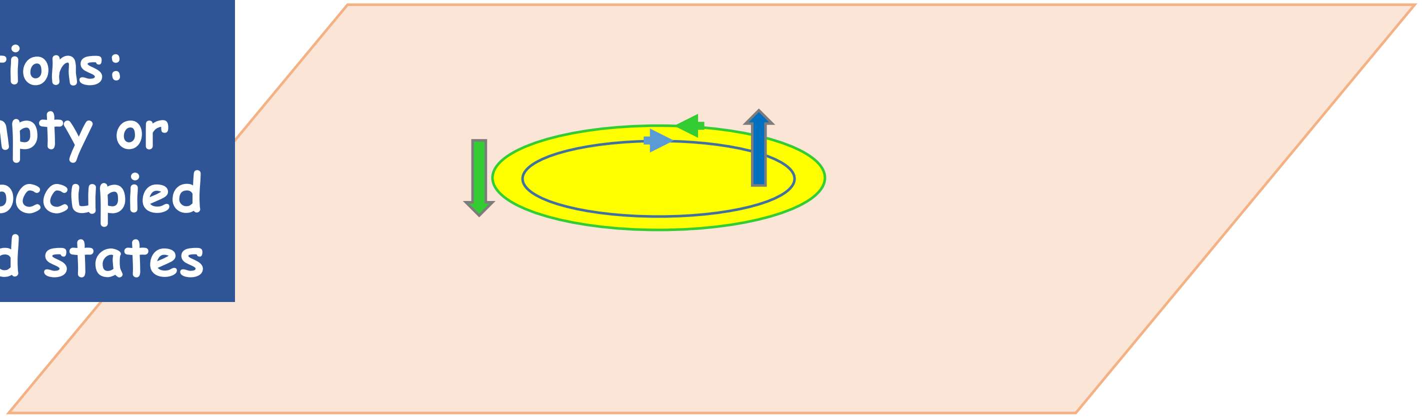
Origin:
1. Chemistry: dangling bonds, etc.
2. Localized energy levels close to the edge



In the presence of disorder the "edge" is not single connected.

Formation of the Kondo Spins

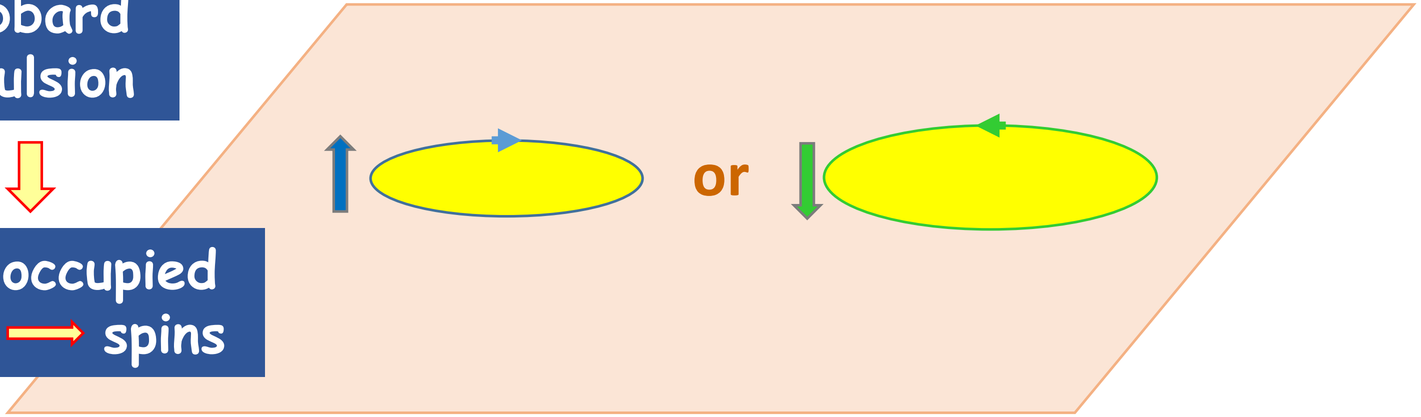
No e-e interactions: Only empty or double occupied localized states



Hubbard repulsion

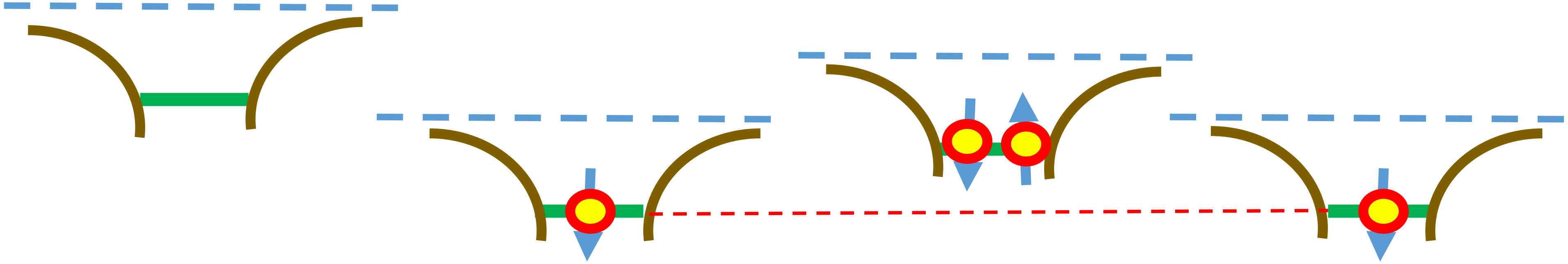
⇩

Single occupied states ⟹ spins

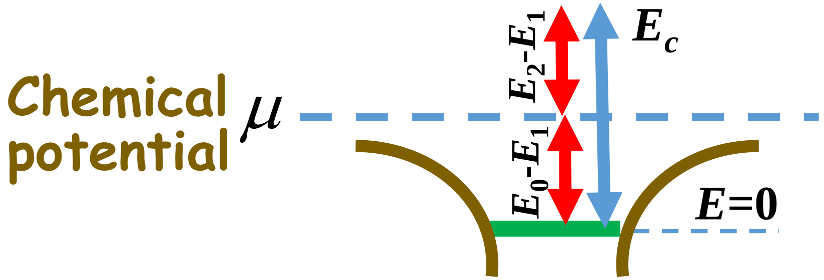


Localized Spins in the presence of itinerant electrons = Kondo Spins

Origin:
1. Chemistry: dangling bonds, etc.
2. Localized energy levels close to the edge



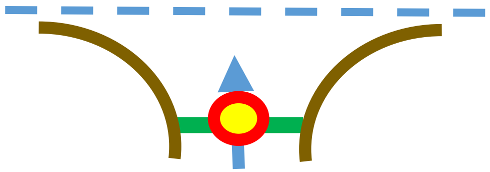
Onsite Hubbard repulsion – Anderson Model
4 states of each localized level. Different energies



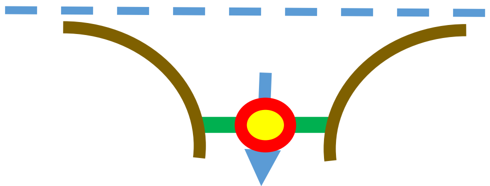
E_c – energy of the repulsion

$$E_c > \mu \Rightarrow \begin{cases} E_0 > E_1 \\ E_2 > E_1 \end{cases}$$

Anderson Model ⇒ Kondo Model

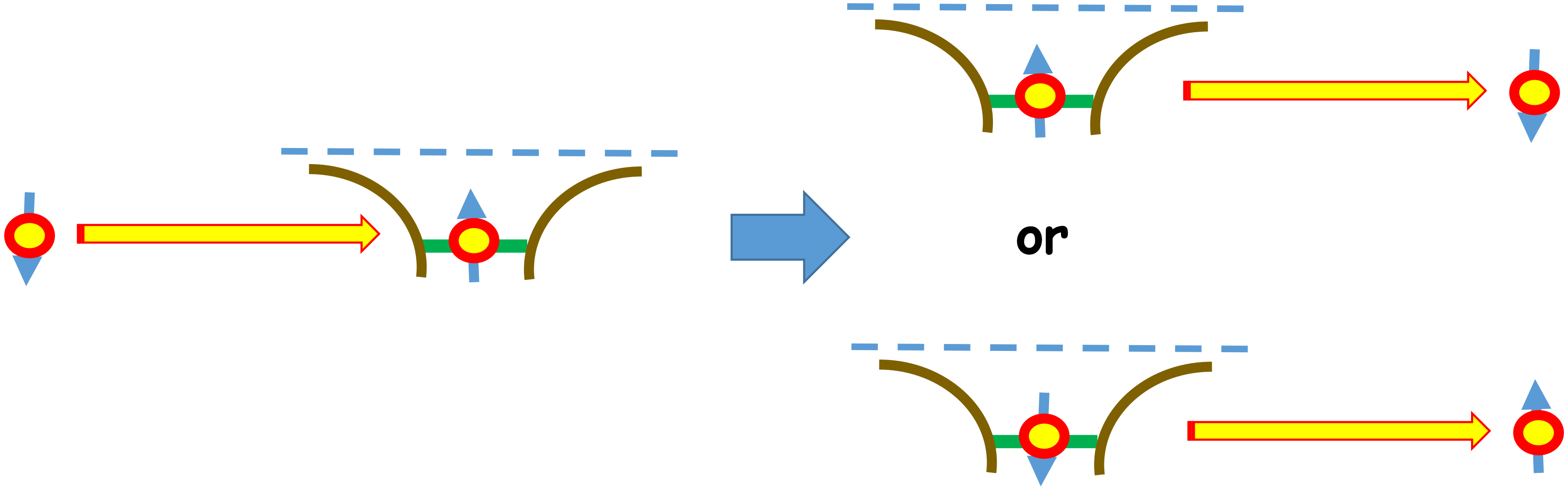


or



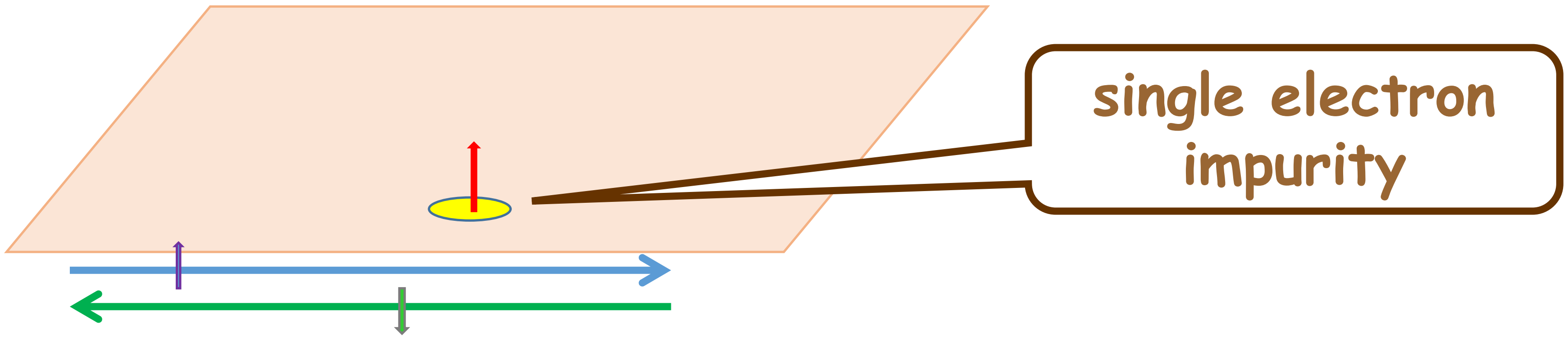
Interaction between the itinerant electrons and the Kondo Spins

$$\hat{H}_{eS} = J\left(\hat{\vec{\sigma}} \bullet \hat{\vec{S}}\right)\delta\left(\vec{r}_e - \vec{r}_S\right) = J\left(\hat{\sigma}_z \bullet \hat{S}_z + \hat{\sigma}^+ \bullet \hat{S}^- + \hat{\sigma}^- \bullet \hat{S}^+\right)\delta\left(\vec{r}_e - \vec{r}_S\right)$$



Kondo Effect – Screening of the localized spins

Magnetic (spin) impurity near the helical edge



Electron-spin interaction:

$$U(1): \quad J_z \sigma^z S^z + J_\parallel (\sigma^x S^x + \sigma^y S^y) =$$

$$= J_z \sigma_z S^z + \frac{J_\parallel}{2} (\sigma^+ S^- + \sigma^- S^+)$$

$\vec{\sigma}$ electron spin

$\vec{S}$ impurity spin

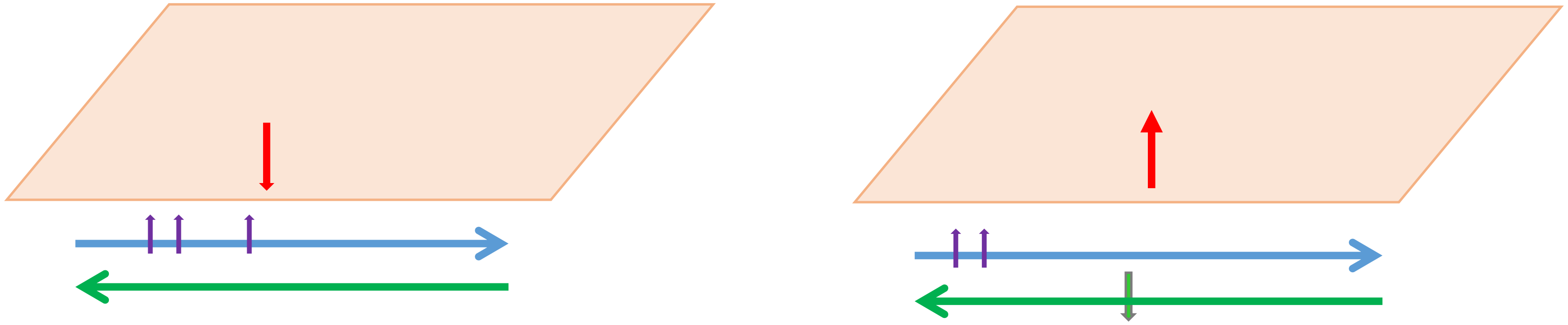
U(1)-symmetry: symmetry under rotation around z-axis in the spin space = conservation of the z-component of the total spin

No influence on the $T = 0$ dc charge transport

Reasons:

$U(1)$ - symmetric (xy-isotropic) electron-spin interaction has no influence on T=0 dc transport

1. Kinematic reason (Tanaka, Furusaki, Matveev (2011))



Spin down impurity can back-scatter a right-moving electron.

However, subsequent backscattering of right-moving electrons is impossible until some left-moving electron reverse the impurity spin!

The impurities can effect ac conductivity but not dc one!

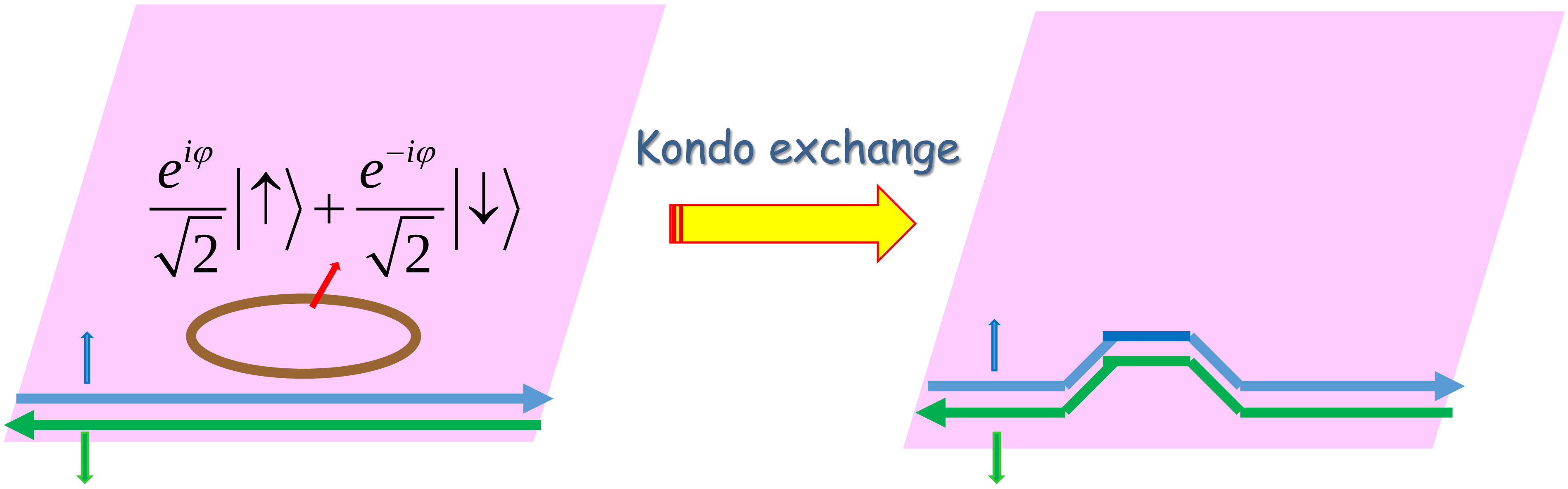
Reason: conservation of S_z

$U(1)$ - symmetric (xy-isotropic) electron-spin interaction has no influence on zero-temperature dc charge transport

Second reason:

Kondo effect - screening of the impurity spin

Recovery of the Time Reversal Symmetry



Does not depend on the U(1)-symmetry

Finite density of spins – competition between the Kondo effect and Ruderman-Kittel-Kasuya-Yosida (RKKY) interaction of the spins

$$H_{S-S} \propto \frac{J^2}{v_F} \sum_{j \neq k} \frac{\vec{S}_j \vec{S}_k \cos[2k_F r_{jk}]}{r_{jk}^{\ d}}$$

d – number of dimensions $\qquad r_{jk} \equiv \left| \vec{r}_j - \vec{r}_k \right|$

Single spin: T-invariance always survives due to the Kondo effect

Finite density of spins: T-invariance can be violated spontaneously (Kondo $\Longleftrightarrow$ RKKY)

but

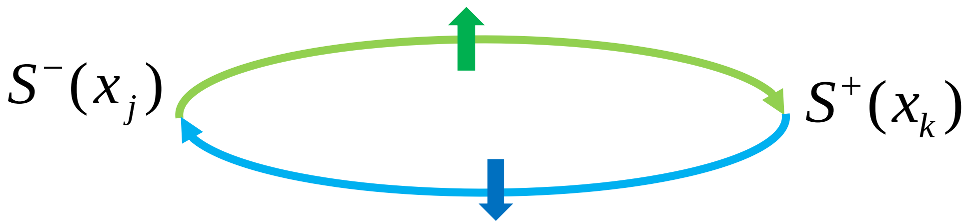
the backscattering would not appear as long as the z-component of the total spin is conserved = the system remains $U(1)$-symmetric, i.e. invariant under rotations in spin space around z-axis.

Q: What if there is a small but finite density of localized and anisotropic spins**?**

$$\cancel{U(1)}: \quad J_z\sigma^z S^z + (J_x\sigma^x S^x + J_y\sigma^y S^y); \qquad J_x \neq J_y$$

Q: What if there is a small but finite density of localized and anisotropic spins?

Indirect exchange of the localized spins – "RKKY" interaction



density of spins →

Kondo — RKKY ⇒ — **Spin glass**

← electron-spin coupling constant

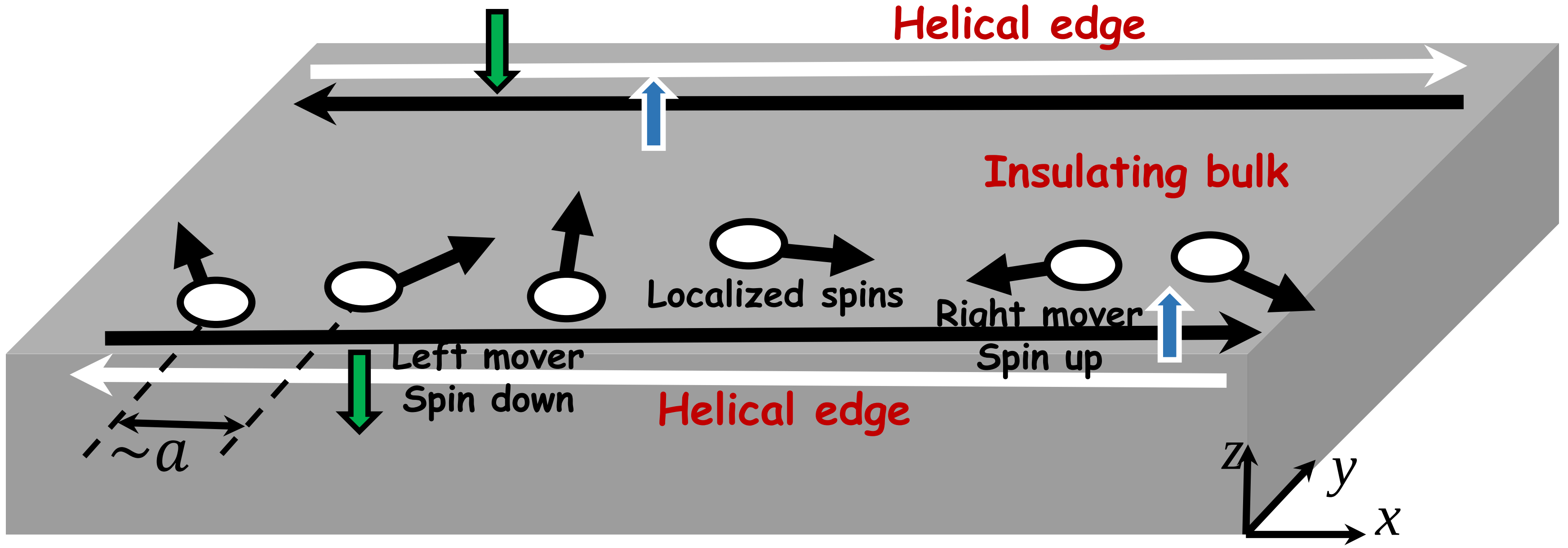
Kondo: T-invariant

Spin glass: Broken T-invariance

Spontaneous breaking of the time-reversal symmetry

	No magnetic anisotropy $U(1)$ S_z^{tot} is conserved	Magnetic anisotropy S_z^{tot} is not conserved ~~$U(1)$~~
No disorder (regular spin chain)	Perfect 1D metal $\sigma = 2e^2/h$	**Band Insulator:** Charged excitations are gapped even at the edge
Disorder	**Goldstone mode** → perfect 1D metal $\sigma = 2e^2/h$	**Anderson Insulator:** Edge states are localized

Model



Localized spin impurities $\vec{S}_j$, located at x_j, $x_{j+1} > x_j$

Linear spin density $\rho_S(x) = \sum_j \delta(x - x_j);$

Averaged spin density: $\rho_S \equiv \langle \rho_S(x) \rangle \equiv \dfrac{1}{a}$

Hamiltonian:

$$H = H_e + H_{e-S}$$

Electron operator:

$$\hat{\Psi}(x) = \frac{1}{\sqrt{2}} \begin{pmatrix} \Psi_{R\uparrow} e^{ik_F x} \\ \Psi_{L\downarrow} e^{-ik_F x} \end{pmatrix}$$

Free helical electrons:

$$H_e = -iv_F \int dx \hat{\Psi}^\dagger(x) \begin{pmatrix} \partial_x & 0 \\ 0 & -\partial_x \end{pmatrix} \hat{\Psi}^\dagger(x) =$$

$$= -iv_F \int dx \left[\Psi^\dagger_{R(\uparrow)}(x) \partial_x \Psi_{R(\uparrow)} - \Psi^\dagger_{L(\downarrow)}(x) \partial_x \Psi_{L(\downarrow)} \right]$$

Electron-spin interaction: $\vec{\sigma}$ - electron spin ($\vec{\sigma} = \hat{\Psi}^\dagger \hat{\vec{\sigma}} \hat{\Psi}$)

$$U(1) : J_z \sigma_z S^z + J_\parallel (\sigma_x S^x + \sigma_y S^y) =$$

$$= J_z \sigma_z S^z + \frac{J_\parallel}{2} (\sigma_+ S^- + \sigma_- S^+)$$

~~$U(1)$~~ : add

$$\frac{\delta J}{2} (\sigma_+ S^+ + \sigma_- S^-)$$

$$J_\parallel = \frac{1}{2}(J_x + J_y), \quad \delta J = \frac{1}{2}(J_x - J_y)$$

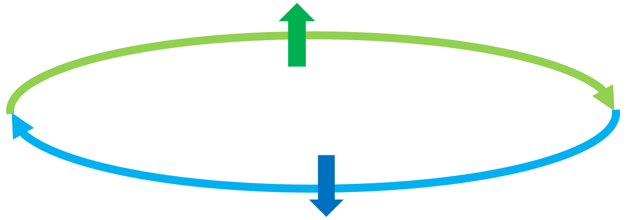
$$|\delta J| << J_\parallel$$

Note: the Hamiltonian is T-invariant

$U(1)$: Electron-spin interaction

$$H_{e-S}=\sum_j\left[J_z S^z\left(\Psi^+_{R(\uparrow)}\Psi_{R(\uparrow)}-\Psi^+_{L(\downarrow)}\Psi_{L(\downarrow)}\right)+\frac{J_\parallel}{2}\left(S_j^+\Psi^+_{L(\downarrow)}\,\mathrm{e}^{2ik_F x_j}\,\Psi_{R(\uparrow)}+S_j^-\Psi^+_{R(\uparrow)}\,\mathrm{e}^{-2ik_F x_j}\,\Psi_{L(\downarrow)}\right)\right]$$

Effective spin-spin ("RKKY") interaction



Helical (chiral) electrons

$$H_{S-S}=-\frac{J_\parallel^2}{8\pi v_F}\sum_{j\neq k}\frac{S_j^+S_k^-\,\mathrm{e}^{2ik_F(x_j-x_k)}+h.c.}{|x_j-x_k|}$$

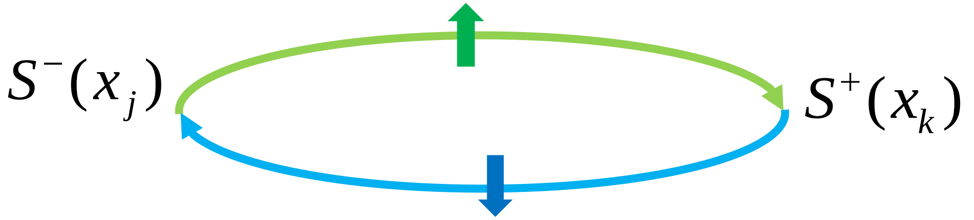
Usual (non-chiral) electrons

$$H_{S-S}\propto\frac{J^2}{v_F}\sum_{j\neq k}\frac{\vec{S}_j\vec{S}_k\cos[2k_F(x_j-x_k)]}{|x_j-x_k|}$$

$U(1)$: Electron-spin interaction

$$H_{e-S} = \sum_j \left[J_z S^z \left(\Psi^+_{R(\uparrow)} \Psi_{R(\uparrow)} - \Psi^+_{L(\downarrow)} \Psi_{L(\downarrow)} \right) + \frac{J_\parallel}{2} \left(S_j^+ \Psi^+_{L(\downarrow)} e^{2ik_F x_j} \Psi_{R(\uparrow)} + S_j^- \Psi^+_{R(\uparrow)} e^{-2ik_F x_j} \Psi_{L(\downarrow)} \right) \right]$$

Effective spin-spin ("RKKY") interaction



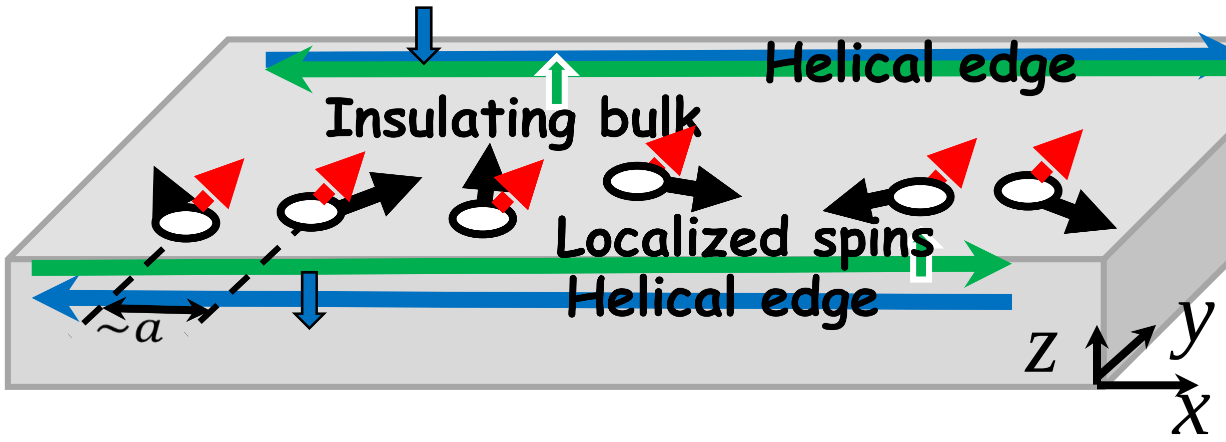
Helical (chiral) electrons	**Usual (non-chiral) electrons**
$H_{S-S} = -\frac{J_\parallel^2}{8\pi v_F} \sum_{j \neq k} \frac{S_j^+ S_k^- e^{2ik_F(x_j - x_k)} + h.c.}{\lvert x_j - x_k \rvert}$	$H_{S-S} \propto \frac{J^2}{v_F} \sum_{j \neq k} \frac{\vec{S}_j \vec{S}_k \cos[2k_F(x_j - x_k)]}{\lvert x_j - x_k \rvert}$

Features of chirality:

1. No $S_j^z S_k^z$ interaction

2. Factorization $e^{2ik_F(x_i - x_j)} = e^{2ik_F x_i} \bullet e^{-2ik_F x_j}$ instead of $\cos[2k_F(x_j - x_j)]$

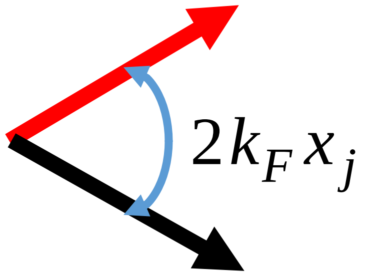
Can be removed by $S_j^\pm \leftarrow \tilde{S}_j^\pm = S_j^\pm e^{\mp 2ik_F x_j}$



$$S_j^{\pm} \rightarrow S_j^{\pm}\, \mathrm{e}^{\mp 2ik_F x_j}$$

black red

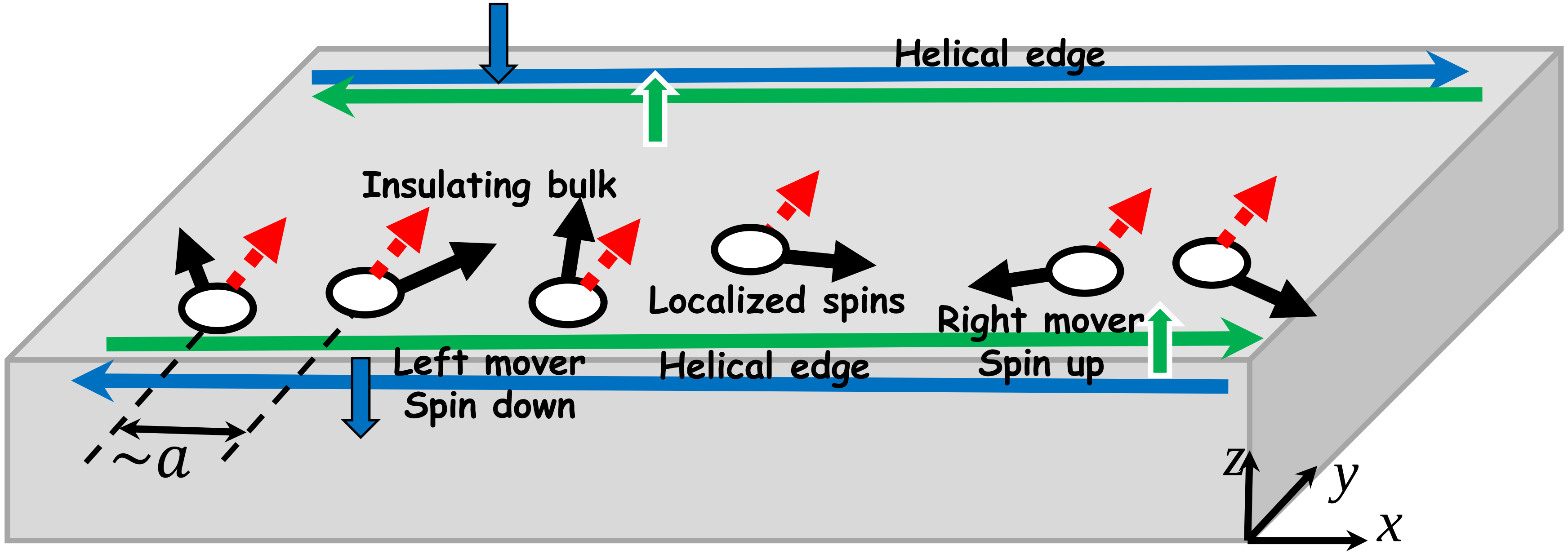
$$\tilde{S}_j^{\pm} = S_j^{\pm} e^{\mp 2 i k_F x_j}$$



Effective spins with ferromagnetic exchange interaction

Ferromagnet (rather than Spin Glass) without mean magnetization

$$H_{S-S}^{(eff)} = -\frac{J_\parallel^2}{8\pi v_F} \sum_{j \neq k} \frac{\tilde{S}_j^+ \tilde{S}_k^- + h.c.}{|x_j - x_k|}$$

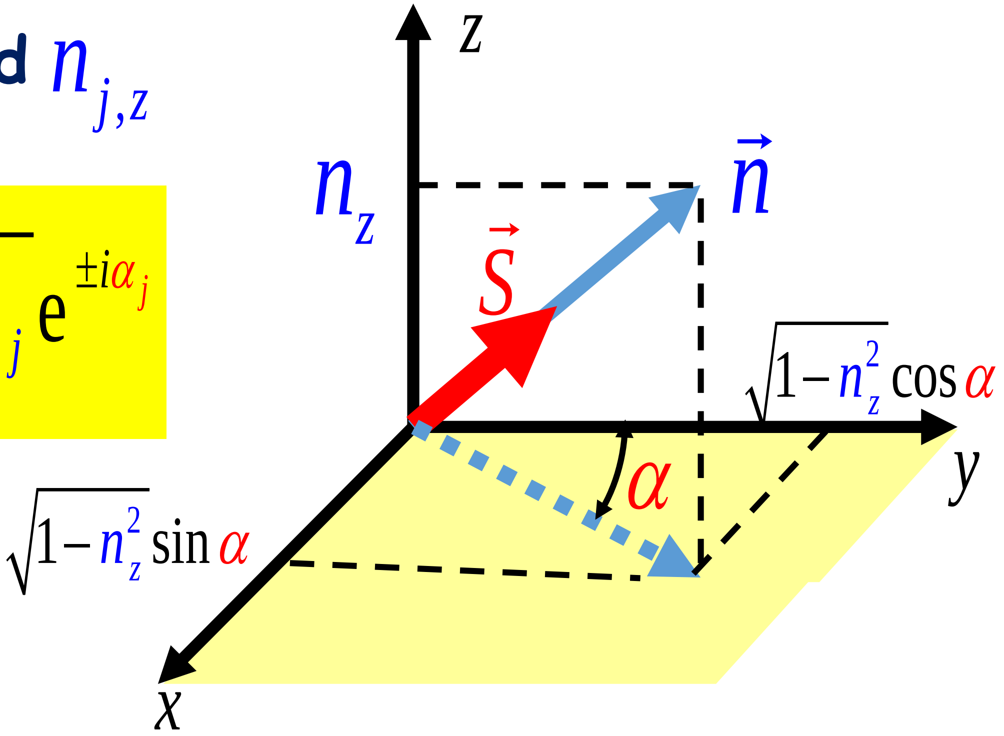


Coherent representation of the spins:

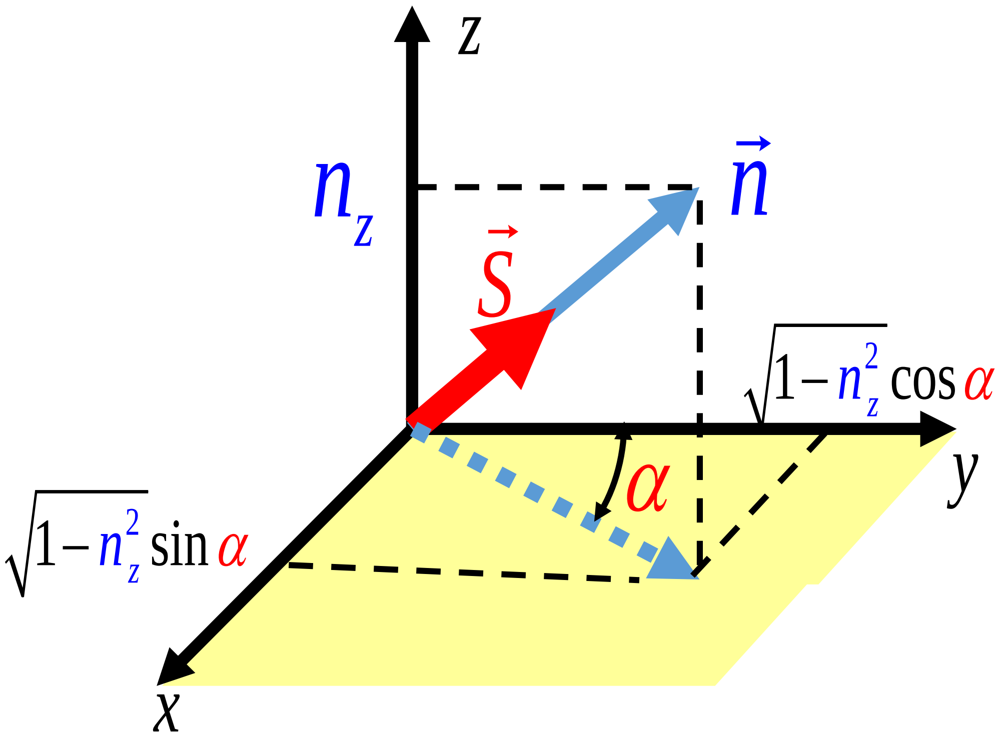
$\vec{S}_j$ **is parametrized by** α_j **and** $n_{j,z}$

$$S_j^z \to \frac{1}{2} n_{z,j}; \quad S_j^{\pm} \to \frac{1}{2}\sqrt{1-n_{z,j}^2}\, e^{\pm i\alpha_j}$$

$$\vec{S}_j \to \frac{1}{2}\vec{n}_j; \quad |\vec{n}_j| = 1$$



Wess, Zumino. (1971).
Witten. (1983)



Need to

1. Integrate over n_z
2. Integrate over $\hat{\Psi}(x)$
3. Take into account the chiral anomaly

$$S[\alpha]=\int \rho_S(x)\,dx\int_0^\beta d\tau\;\frac{1}{2\pi K}\left\{\frac{1}{u}(\partial_\tau\alpha)^2+u(\partial_x\alpha)^2\right\}$$

$$K=\frac{u}{v_F}$$

$$u=\frac{J_\parallel}{2\pi}\sqrt{\ln\left(\frac{E_B}{\Delta}\right)}$$

Mapping for isotropic exchange

Free chiral edge electrons
El-spin exchange, $U(1)$ -symmetry

⇓

Fermions acquire a gap;
A bosonic mode remains gapless

⇓

Luttinger Liquid with small velocity and small Luttinger parameter

⇓

Ideal Metallic Conductance in the presence of a potential disorder

$$u = \frac{J_{\parallel}}{2\pi}\sqrt{\ln\left(\frac{E_B}{S\rho_S J_{\parallel}}\right)} \ll v_F$$

Effective velocity

$$K = \frac{u}{v_F} \ll 1$$

Luttinger parameter

$$\rho_{el} = \frac{-1}{2\pi}\partial_x \alpha$$

Bosonic field
Goldstone mode

Mapping for anisotropic exchange

Electron-spin interaction: $\frac{J_\parallel}{2}(\sigma_j^+ S_j^- + \sigma_j^- S_j^+) + \frac{\delta J_j}{2}(\sigma_j^+ S_j^+ + \sigma_j^- S_j^-)$

$\varepsilon_j \propto \frac{\delta J_j}{J_\parallel} e^{4ik_F x_j}$

Anisotropy of j-th spin

If ε_j is independent of j, then the spectrum is **gapfull**

More likely ε_j is random

$\langle \varepsilon_k^* \varepsilon_j \rangle = d\delta_{jk}$

Measure of the U(1) breaking disorder

Continuous limit

$\varepsilon_j \Leftarrow \varepsilon(x) \qquad \langle \varepsilon(x)\varepsilon^*(x') \rangle = \rho_S d\delta(x - x')$

Matsubara Action (broken $U(1)$-symmetry)

Electron-spin interaction:

$$\frac{J_{\parallel}}{2}(\sigma_j^+ S_j^- + \sigma_j^- S_j^+) + \frac{\delta J_j}{2}(\sigma_j^+ S_j^+ + \sigma_j^- S_j^-)$$

$$\varepsilon_j \propto \frac{\delta J_j}{J_{\parallel}} e^{4ik_F x_j}$$

Modified boson action:

$$S[\alpha] \rightarrow \int d\tau dx \left[\frac{v_F}{8\pi u^2}[(\partial_\tau \alpha)^2 + u^2(\partial_x \alpha)^2] + \mathrm{Re}\left[\varepsilon(x) e^{2i\alpha(x,\tau)}\right] \right]$$

Mapping for anisotropic exchange

$$S[\alpha] \rightarrow \int d\tau dx \left[\frac{v_F}{8\pi u^2} [(\partial_\tau \alpha)^2 + u^2 (\partial_x \alpha)^2] + \mathrm{Re}\left[\varepsilon(x) e^{2i\alpha(x,\tau)} \right] \right]$$

$$\varepsilon_j \propto \left(\delta J_j / J_\parallel\right) \exp\left(4ik_F x_j\right)$$

$$\langle \varepsilon^*(x)\varepsilon(x') \rangle = \rho_S d\delta(x - x')$$

Mapping on the problem of the pinning of one-dimensional charge-density wave by potential disorder. (Giamarchi & Schulz, 1988)

Localization length

$$L_{loc} = a \left(\frac{v_F}{J_\parallel} \right)^{\frac{2-2K}{3-2K}} \left[d \ln \left(\frac{aE_B}{J_\parallel} \right) \right]^{\frac{1}{2K-3}}$$

$$u = \frac{J_\parallel}{2\pi} \left(\ln \frac{aE_B}{J_\parallel} \right)^{1/2} << v_F$$

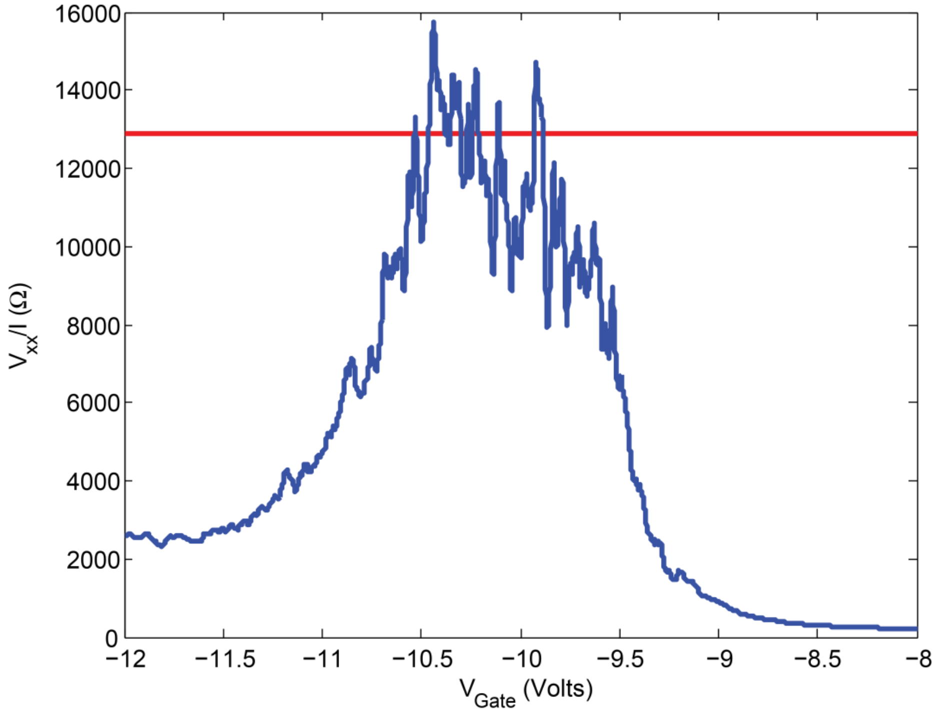
$$K = \frac{u}{v_F} \ll 1$$

$$L_{loc} = a\left(\frac{v_F}{J_{\|}}\right)^{\frac{2-2K}{3-2K}}\left[d\ln\left(\frac{aE_B}{J_{\|}}\right)\right]^{\frac{1}{2K-3}}$$

$$u = \frac{J_{\|}}{2\pi}\left(\ln\frac{aE_B}{J_{\|}}\right)^{1/2} << v_F$$

$$K = \frac{u}{v_F}$$

Localization length can be of the order or even larger than the system size ⇨ something like a quantization



Mesoscopic fluctuations are probably caused by the rearrangement of the spins with the change of the gate voltage

Conclusions

Interaction of helical edge electrons with closely located spin (Kondo) impurities lead to Anderson localization of electrons *if the total z-component of spin is not conserved (broken* $U(1)$ *symmetry)*

Physical interpretation:

1. In the presence of $U(1)$ symmetry spins rotate in the xy-plane. This restores effectively the time reversal symmetry and prevents localization;

2. If $U(1)$ symmetry is randomly broken, the spins are pinned, which means a spontaneous breaking of T-invariance, i.e. there remains no protection against Anderson localization

Q: Is the topological insulator a **distinct** new state of matter **?**

Q: Is the topological insulator a distinct new state of matter or it is a conventional Anderson insulator with the localization length at the edge can substantially exceed the one in the bulk. **?**

Note that an arbitrary small concentration of Kondo impurities with arbitrary weak anisotropy eventually destroys the quantization.

Albeit electrons at 1d helical edge of a 2D topological insulator are more protected from an influence of random imperfections, they are still subjected to Anderson localization similar to the usual 1D conductors