Anderson Localization – Looking Forward

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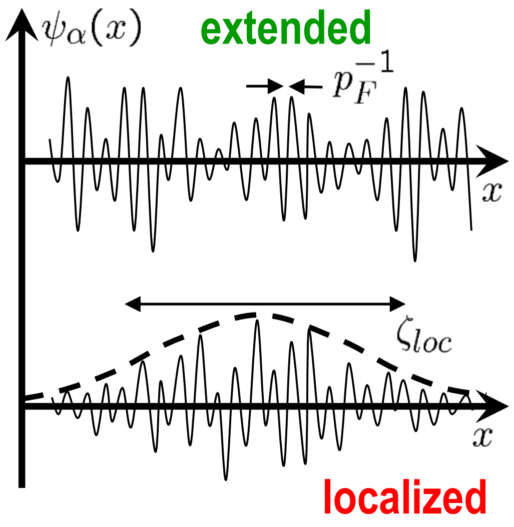
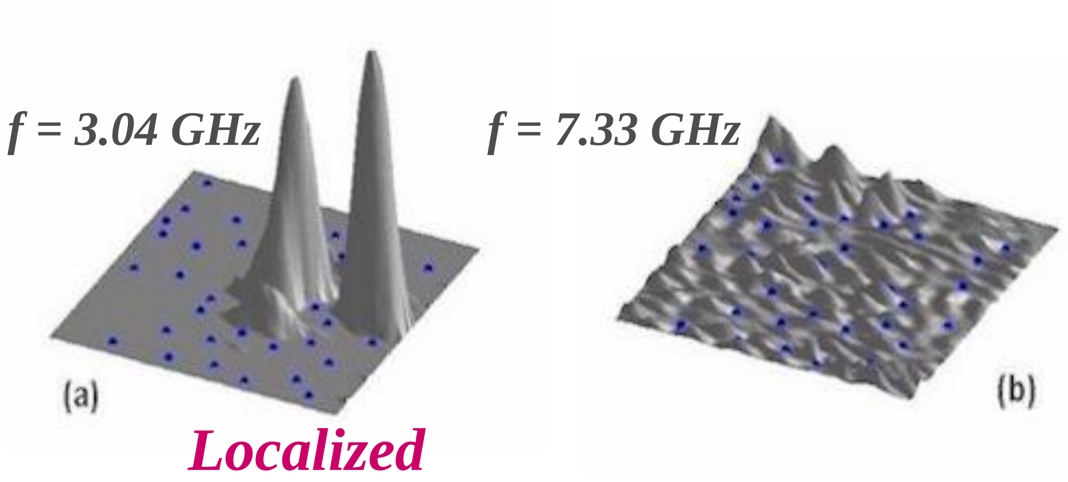
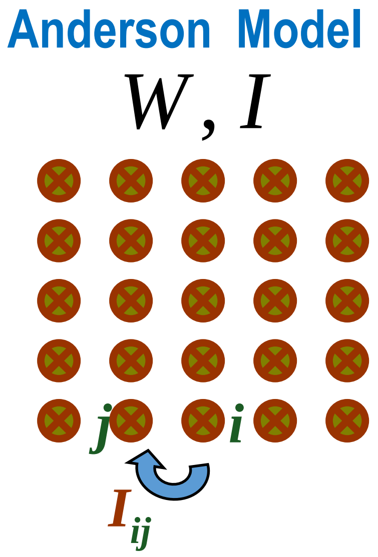
Lecture3
September, 10, 2015

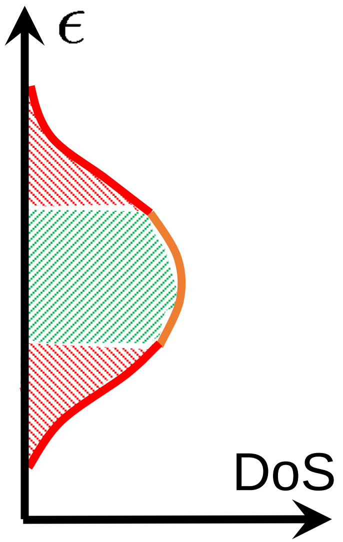
Outline

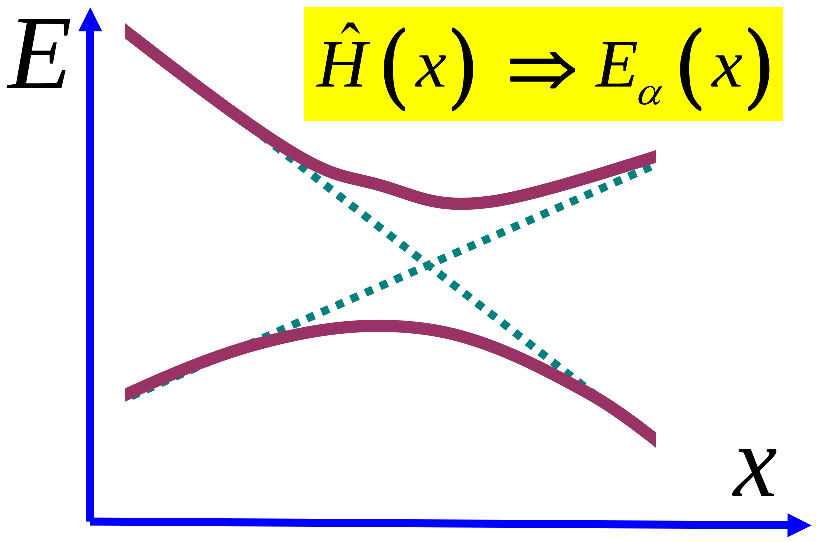
1. Introduction
2. Anderson Model; Anderson Metal and Anderson Insulator
3. Localization beyond the real space. Integrability and chaos.
4. Spectral Statistics and Localization
5. Many-Body Localization.
6. Many-Body Localization of the interacting fermions.
7. Many-Body localization of weakly interacting bosons.
8. Many-Body Localization and Ergodicity

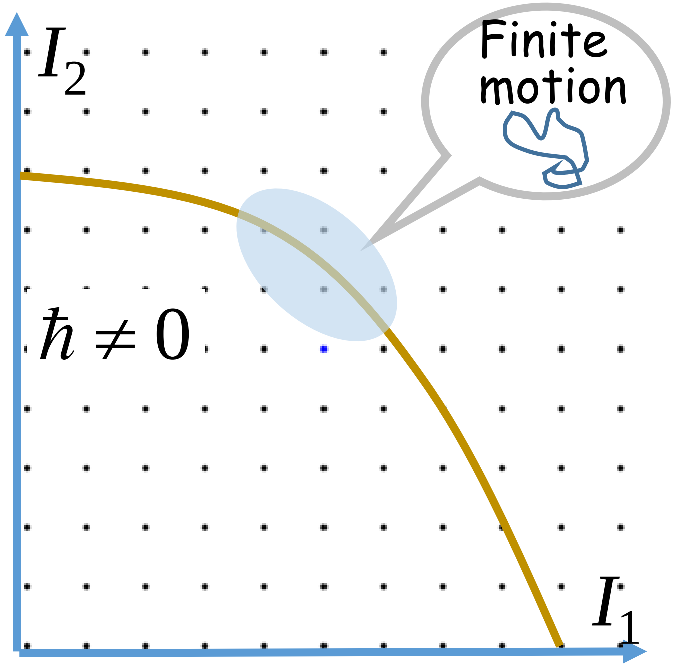

$\psi_\alpha(x)$
extended
p_F^{-1}
x
ζ_{loc}
x
localized

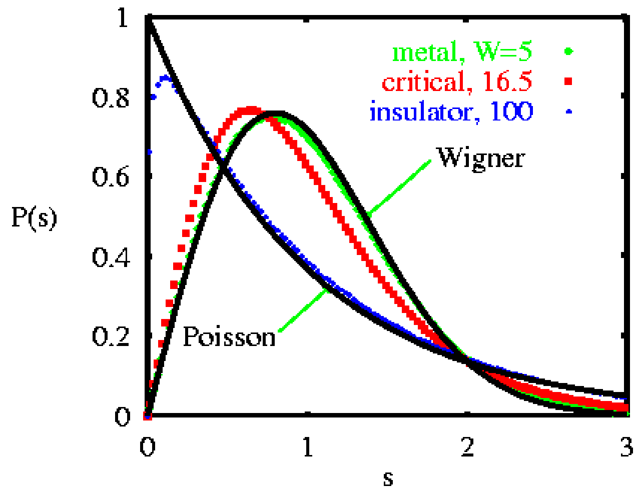

f = 3.04 GHz
f = 7.33 GHz
(a)
(b)
Localized

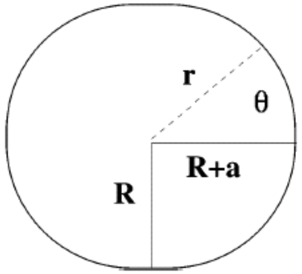

Anderson Model
W, I
j
i
I_{ij}

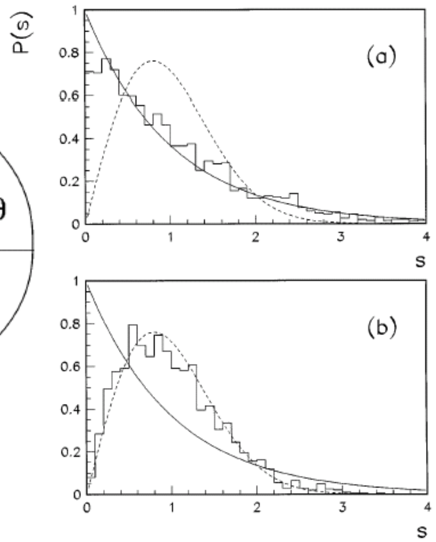

ϵ
DoS


E
$\hat{H}(x) \Rightarrow E_\alpha(x)$
X

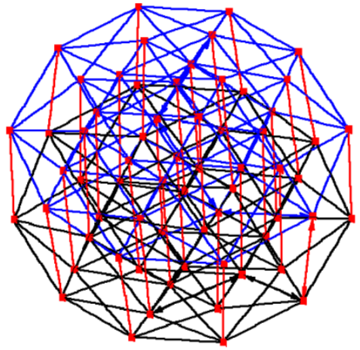

I_2
Finite motion
$\hbar \neq 0$
I_1

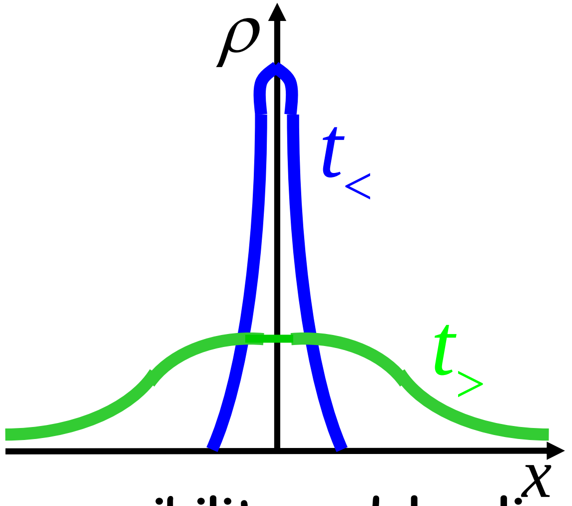




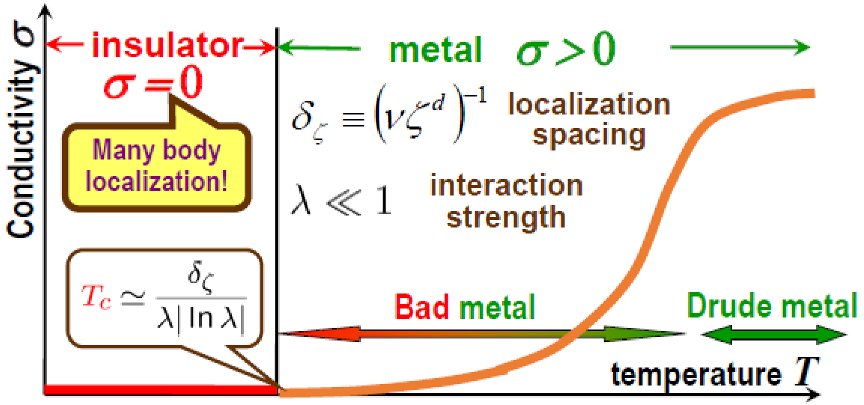


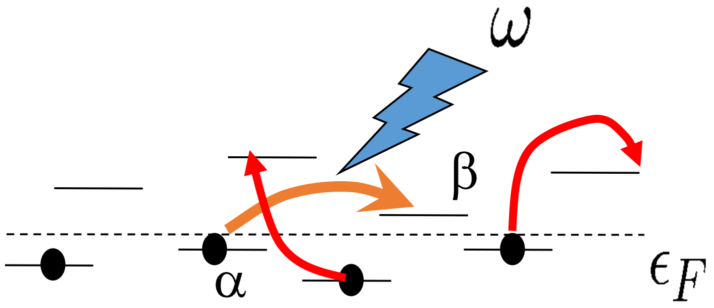
$$\hat{H} = \sum_{i=1}^{N} B_i \hat{\sigma}_i^z + \sum_{i \neq j} J_{ij} \hat{\sigma}_i^z \hat{\sigma}_j^z + I \sum_{i=1}^{N} \hat{\sigma}_i^x \equiv \hat{H}_0 + I \sum_{i=1}^{N} \hat{\sigma}_i^x$$





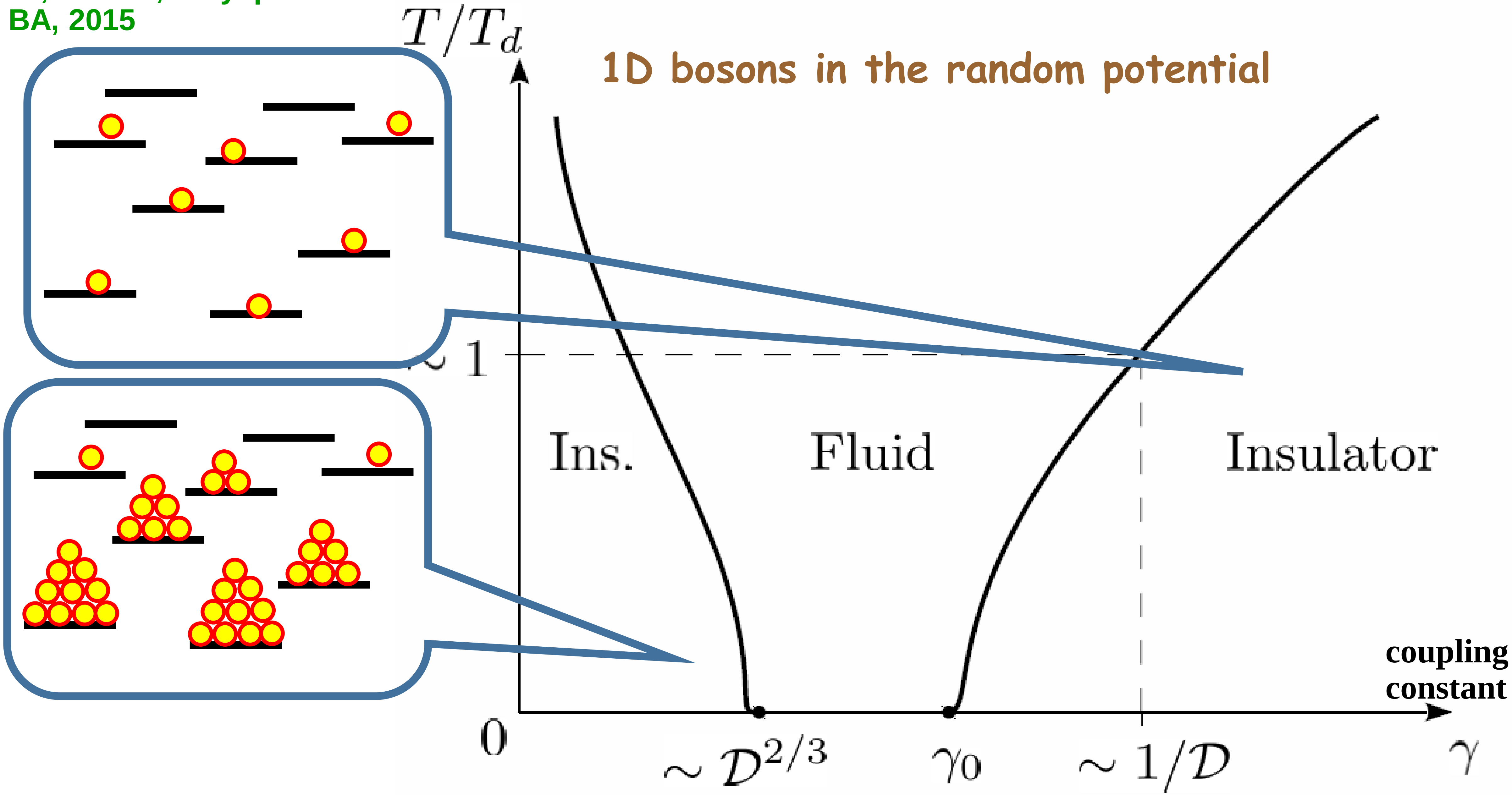
Irreversibility = delocalization





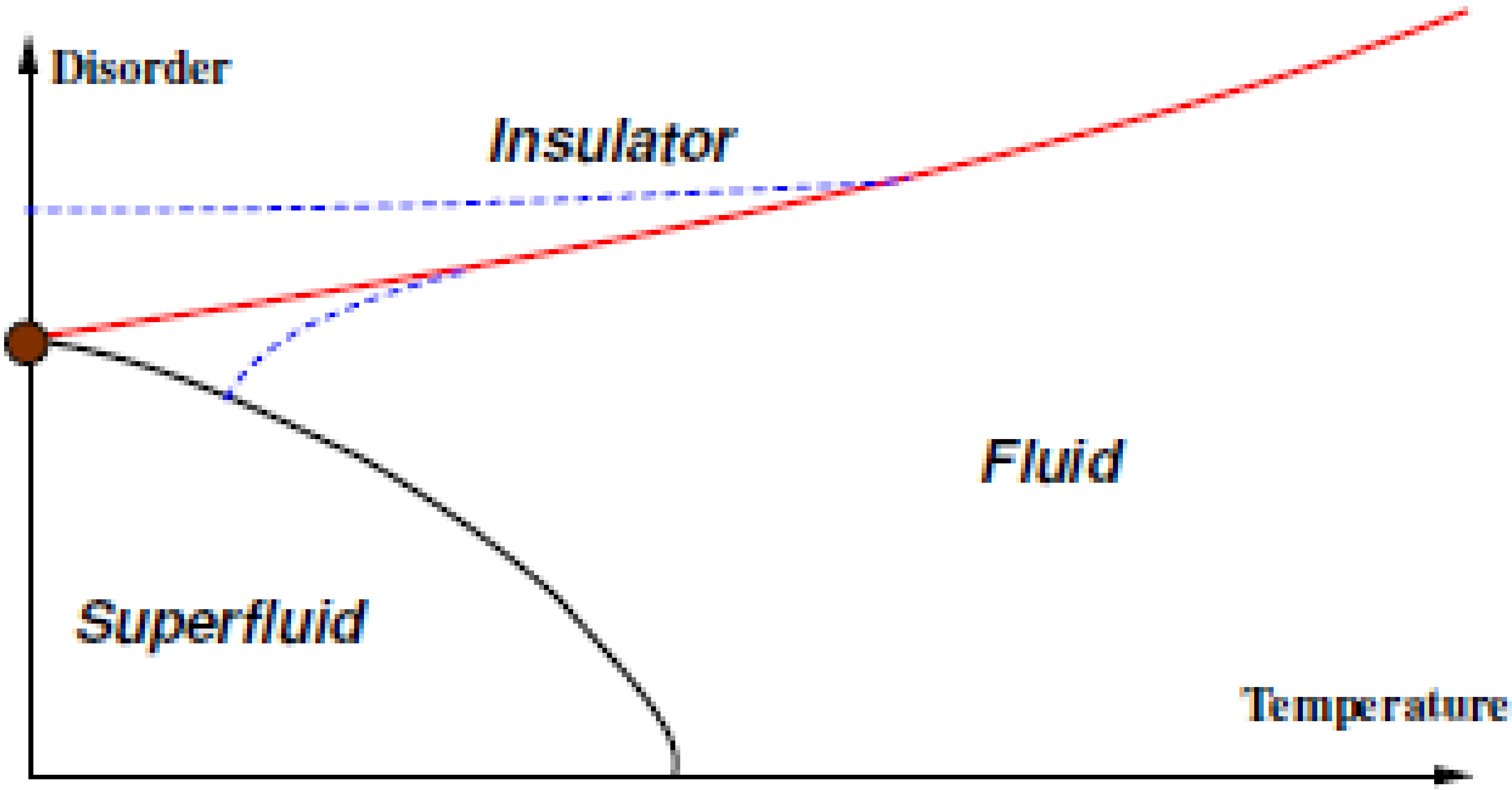
Michal, Aleiner, Shlyapnikov and BA, 2015

1D bosons in the random potential

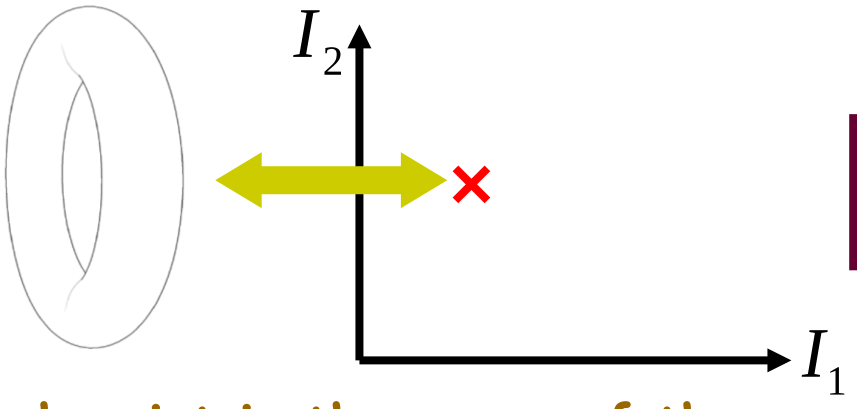


Two transitions

Disordered interacting bosons in two dimensions

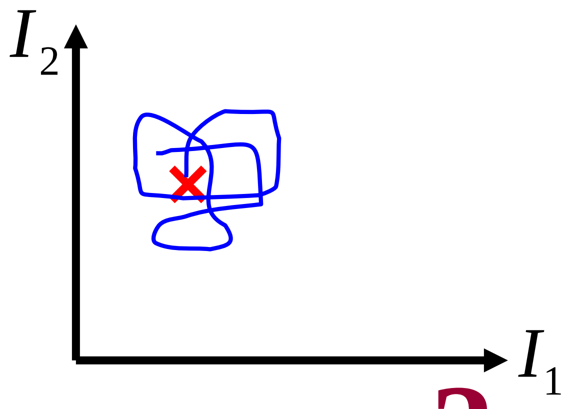


Arnold diffusion



Each point in the space of the integrals of motion corresponds to a torus and vice versa

$\hat{V} \neq 0$



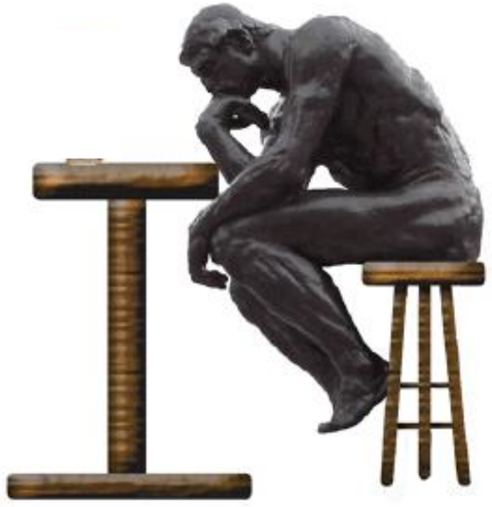
Finite motion?

$$d = 2$$

All classical trajectories correspond to a finite motion

When a theorist is asked to evaluate the stability of a table with 4 legs he/she

1. Evaluates the stability of a table with 1 leg, then
2. Evaluates the stability, of a table with infinite number of legs and after that
3. Spends the rest of the life in attempts to evaluate the stability of the table with an arbitrary number of legs.

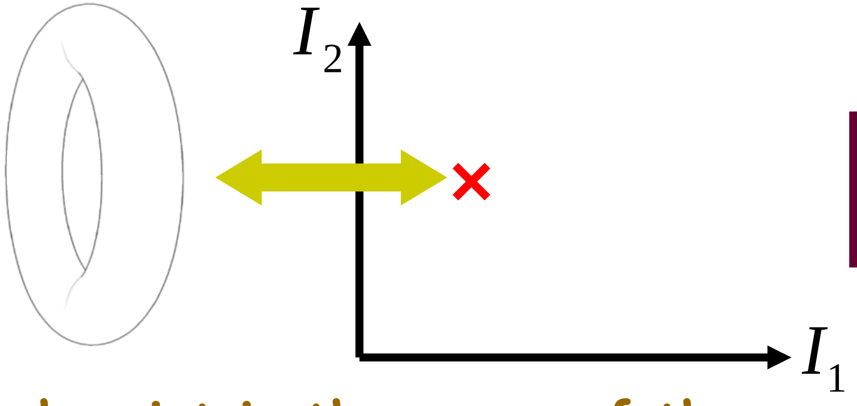


When a mathematician is asked to evaluate the stability of a table with 4 legs he/she

1. Evaluates the stability of a table with 1 leg, then
2. ~~Evaluates the stability, of a table with infinite number of legs and after that~~
3. Spends the rest of the life in attempts to evaluate the stability of the table with an arbitrary number of legs.

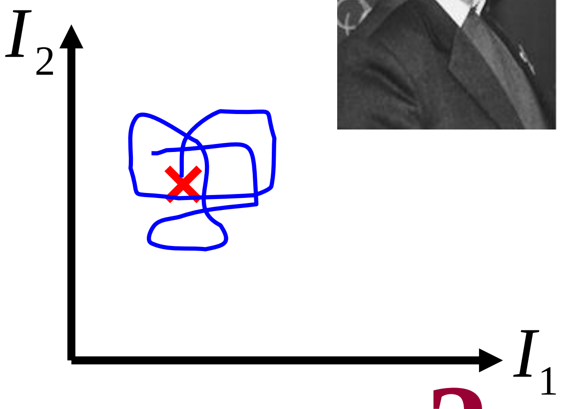
?

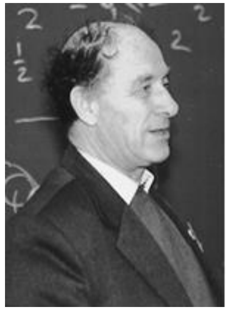
Arnold diffusion



Each point in the space of the **integrals of motion** corresponds to a torus and vice versa

$\hat{V} \neq 0$





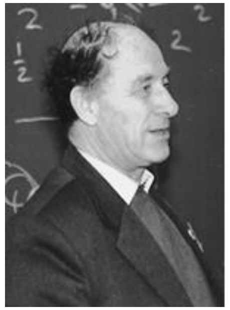
Finite motion?

$d = 2$ — All classical trajectories correspond to a finite motion

$d > 2$ — Most of the trajectories correspond to a finite motion

However small fraction of the trajectories goes infinitely far

Arnold diffusion

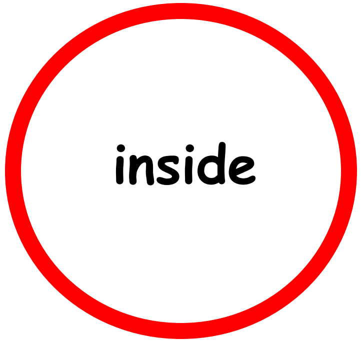


1. Most of the tori survive – KAM
2. Classical trajectories do not cross each other

space	# of dimensions
real space	d
phase space	$2d$
energy shell	$2d$-1
tori	d

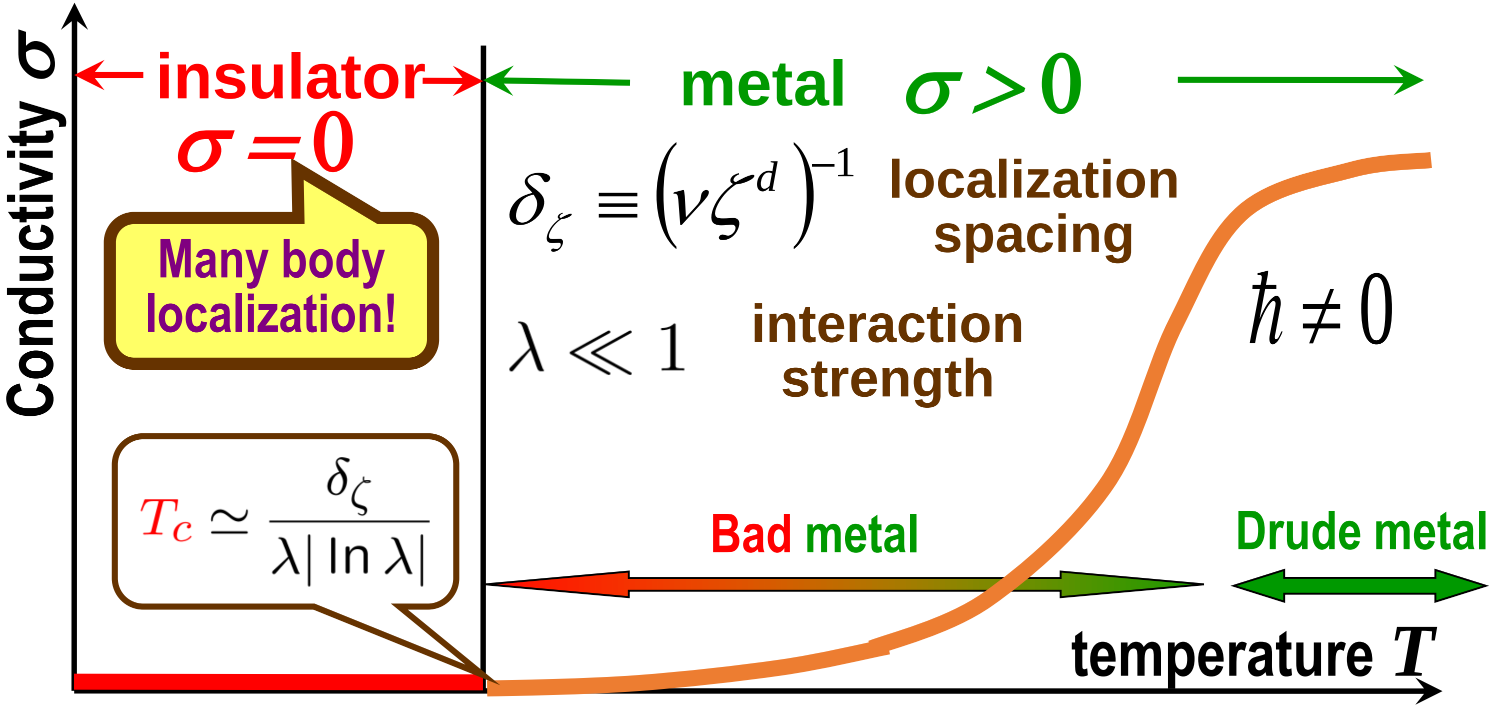
$$d = 2 \quad \Rightarrow \quad d_{en.shell} - d_{tori} = 1$$

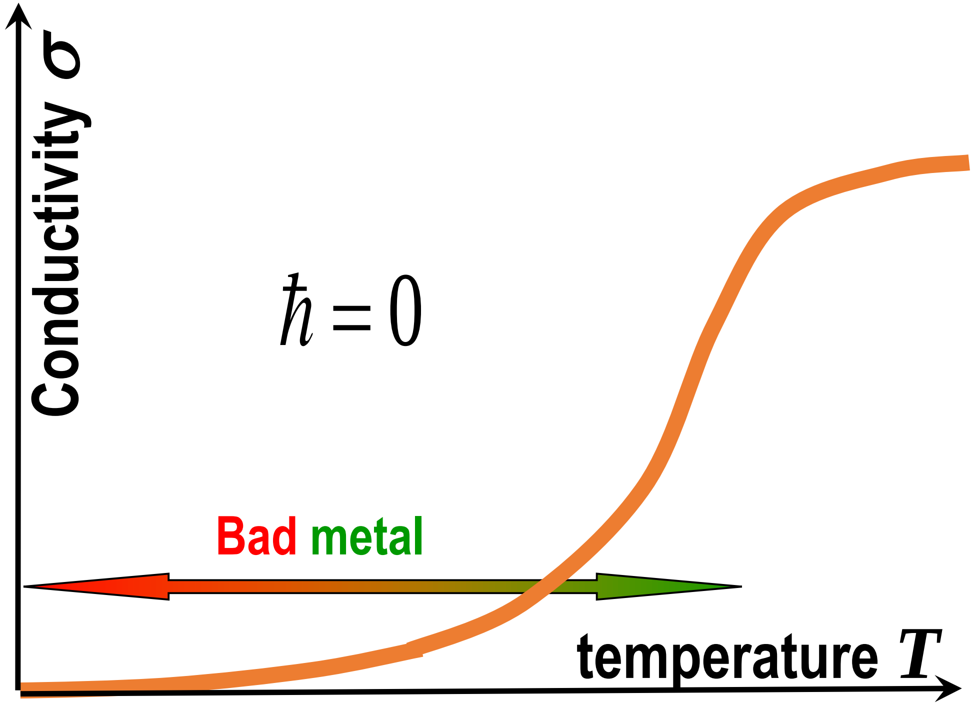
Each torus has “inside” and “outside”



$$d = 2 \quad \Rightarrow \quad d_{en.shell} - d_{tori} = 1$$

A torus does not have “inside” and “outside” as a ring in >2 dimensions





Q: What happens in the classical limit $\hbar \to 0$?

Speculations: 1. No transition $T_c \to 0$
2. Bad metal still exists

Reason: Arnold diffusion

Large number d of the degrees of freedom

Conventional Boltzmann-Gibbs Statistical Physics

Equipartition Postulate

Ergodicity: time average = space (ensemble) average

Chaos $H(\{p_i, q_i\})$

Hamiltonian

?

$$H = H_0 + \lambda V$$

Integrable Systems

d degrees of freedom

d integrals of motion

Ergodicity is violated

Invariant tori dimension d

Hamiltonian $H_0(\{p_i, q_i\})$

Energy shell, dimension $2d - 1$

Classical Dynamics

λ ← Ergodicity / Equipartition — Fermi, Pasta, Ulam system (connected nonlinear oscillators) — KAM region / Arnold diffusion / Non-ergodic — 0

Solar system

. . .

Quantum Dynamics ???

Anderson Localization:

One quantum particle in a random potential

- Strong enough disorder – the eigenstates are localized
- Weak disorder – maybe the eigenstates are extended
- Localization – Delocalization – in real space

- Not only quantum dynamics – any wave dynamics

Many-Body Localization:

Isolated quantum system, many degrees of freedom

- Close to the integrability – the eigenstates are localized
- Far from the integrability –the eigenstates are extended
- Localized – Extended: – space of quantum numbers $\{\mu\}$

- Genuine quantum phenomenon

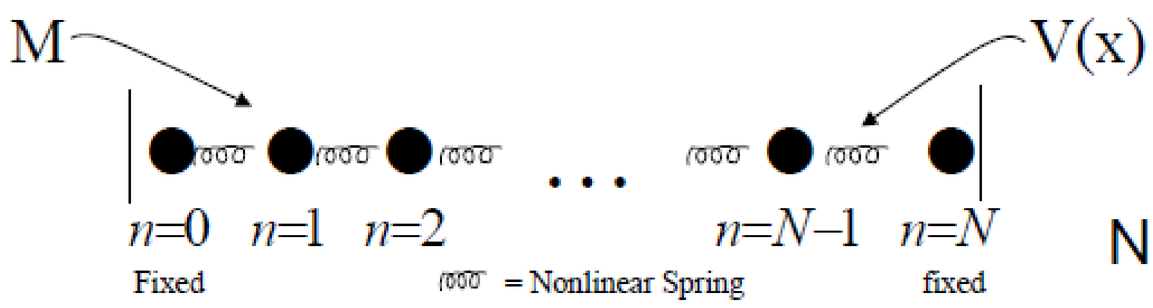
Classical Dynamical Systems:

Are the dynamics ergodic outside the KAM regime?

For some low-dimensional systems one can prove the ergodicity: Sinai billiard, Bunimovich billiard, etc.

At least some systems with high number of dimensions are known to be non-ergodic:

- Solar System
- Fermi-Pasta-Ulam system of connected non-linear oscillators
- .. .

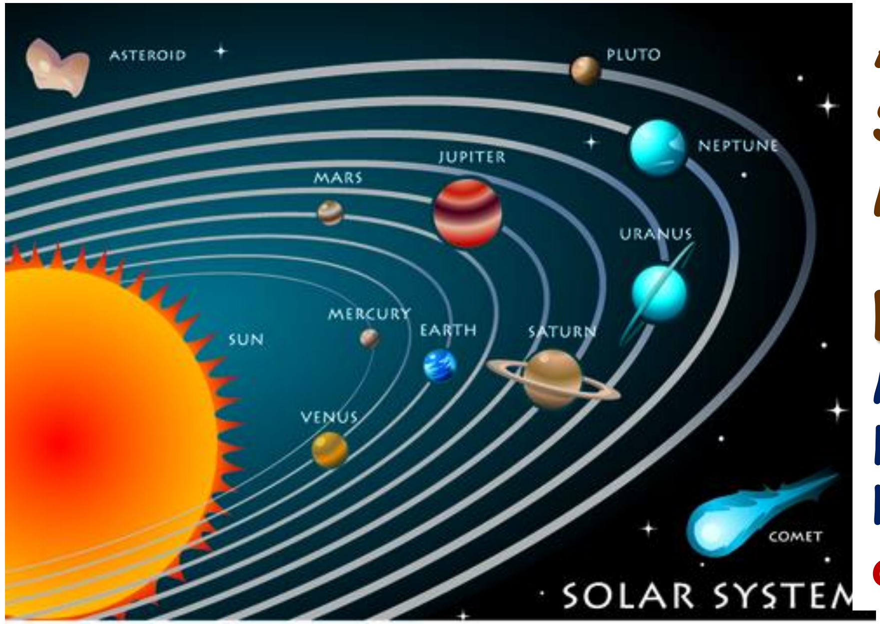


N=32, 64

$$V(x) = \tfrac{1}{2} kx^2 + \alpha/3\, x^3 + \beta/4\, x^4$$

“The results of the calculations (performed on the old MANIAC machine) were interesting and quite surprising to Fermi. He expressed to me the opinion that they really constituted a little discovery in providing limitations that the prevalent beliefs in the universality of “mixing and thermalization in *non-linear* systems may not always be justified.”

[S.Ulam]



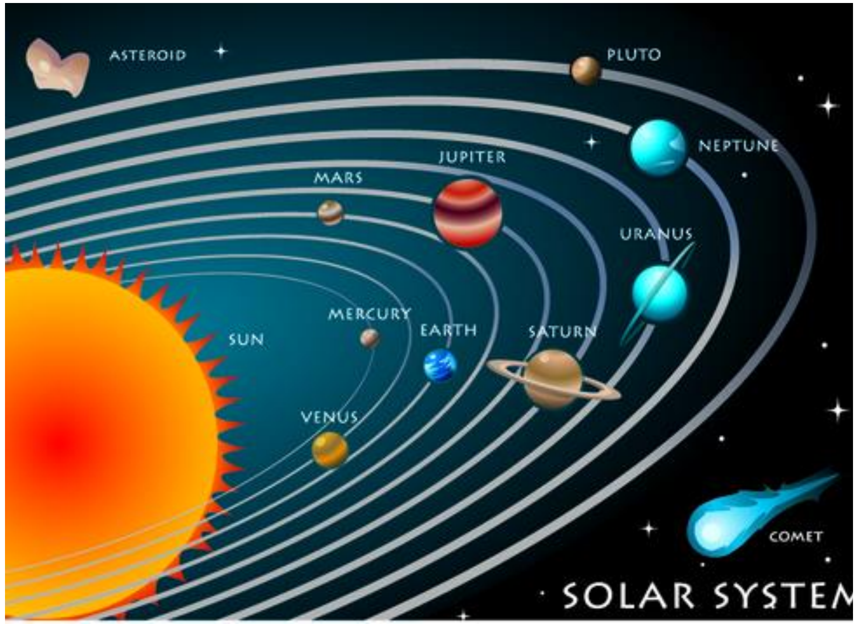
Age: ~4.5 Billion years
Sun dies in ~8 Billion years
Mass 1.0014 Solar masses

Newton:
Motion of a single planet around the Sun. However, there are 8 planets (Newton knew 6). Each one exerts forces on the others – small and periodically varying,.

Newton: "…the Planets move one and the same way in Orbs concentric, some inconsiderable Irregularities excepted, which may have arisen from the mutual Actions of Comets and Planets upon one another, and which will be apt to increase, till this System wants a Reformation.",

God has to intervene continuously to stabilize the world?!

Leibniz sneered at Newton's conception, as being that God so incompetent as to be reduced to miracles in order to rescue his machinery from collapse.



Age: ~4.5 Billion years
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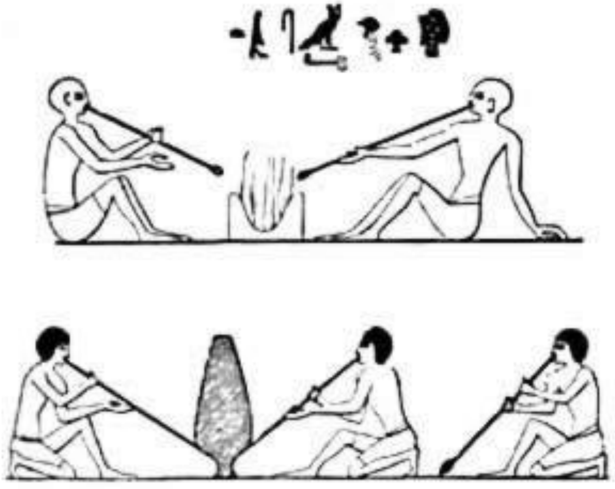
Isaac Newton:

Motion of a single planet around the Sun. However, there are 8 planets (Newton knew 6). Each one exerts small and periodically varying forces on the others

- The positions of the planets in $>10^8$ years are unpredictable: they are too sensitive to initial condition - chaos.
- In 8 billion years (just before the Sun dies) the orbits will most likely be similar to their present ones.
- The unpredictability is mostly in the orbital phases, collisions between planets are unlikely in spite of the chaos.
- Ensemble of solar systems with slightly different parameters at the present time (random shifts ~1mm): ~1% percent probability that Mercury collides with Venus before the death of the Sun.

The solar system is neither absolutely stable nor ergodic

Glassy States of Matter:



Glass in Egyptian tombs – no tendency for ordering/thermalization in ~3000 years

Ideal (no disorder) 1D Josephson array

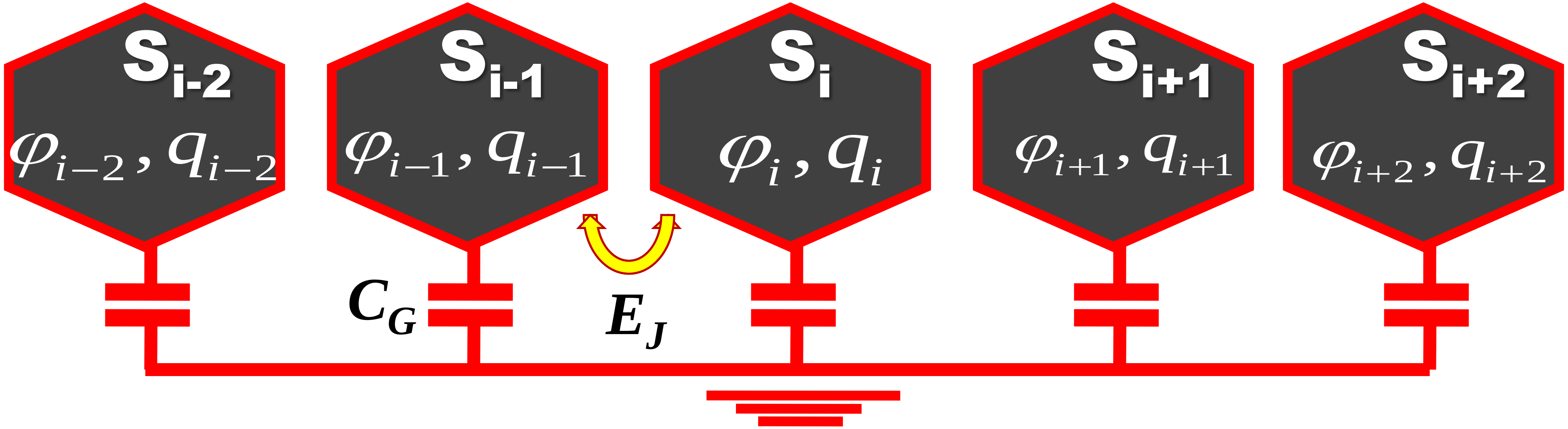
Superconducting island # ***i***

$\Delta \rightarrow \infty$

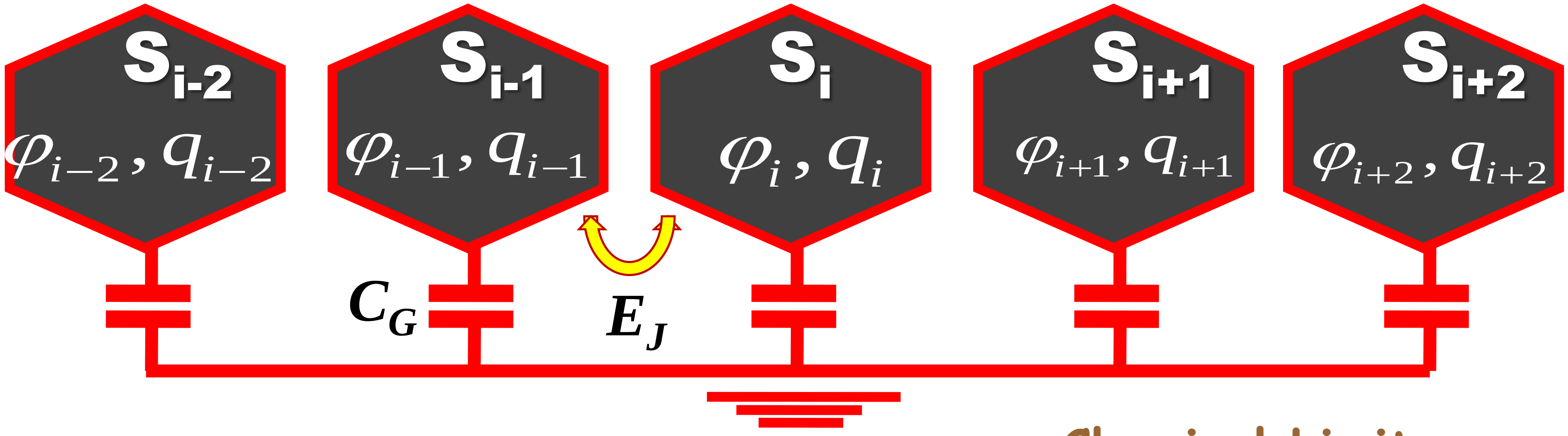
φ_i Phase of the order parameter

q_i Electric charge in units of **2e**

Canonically conjugated variables: $\left[\varphi_j , q_k\right] = i\delta_{jk}$



Ideal (no disorder) 1D Josephson array



E_J Josephson energy

$E_c = 1/C_G$ Charging energy

Classical Limit

$$\frac{E_c}{E_J} \to 0 \qquad q = \frac{d\varphi}{dt} \to 0$$

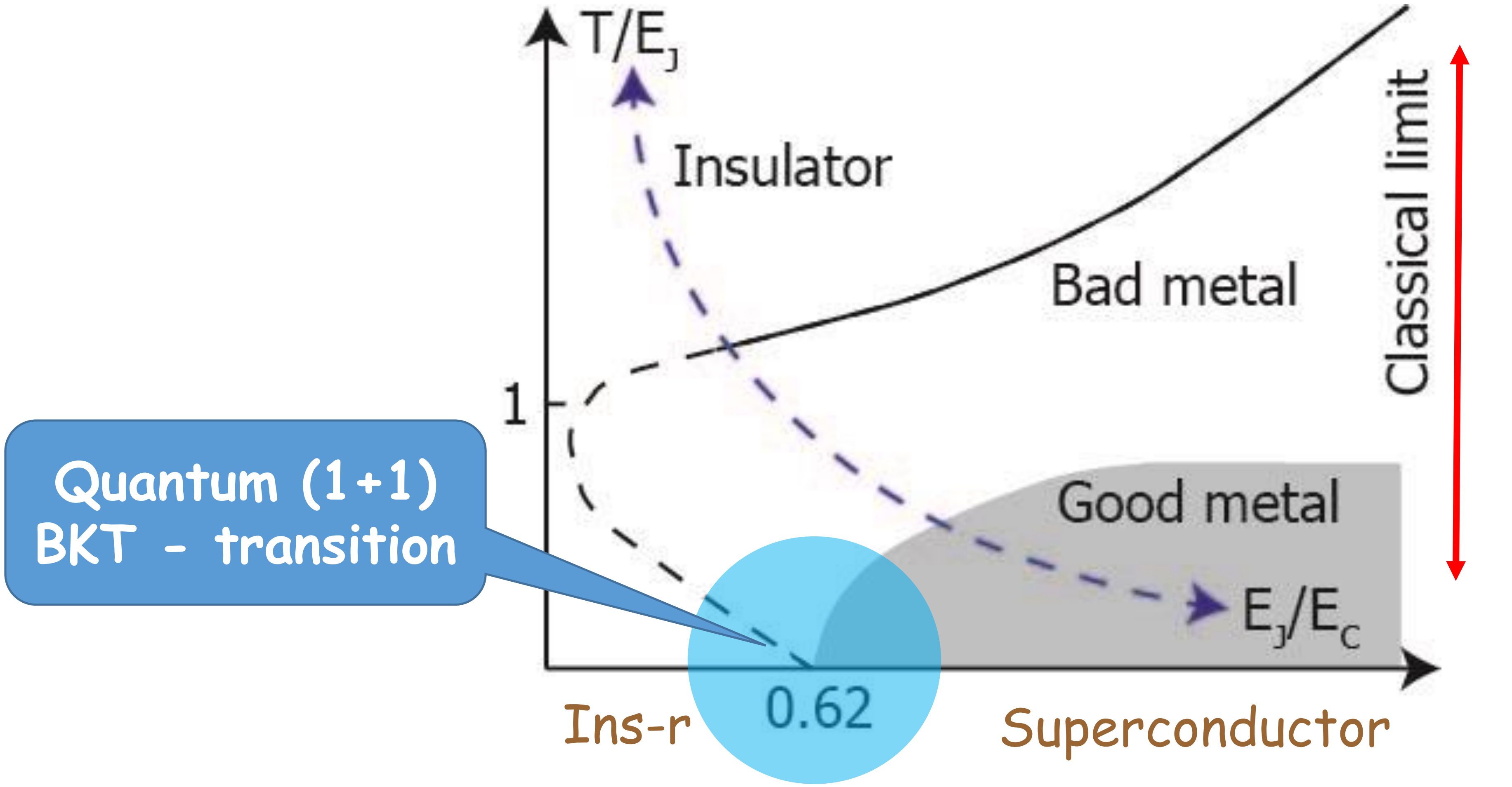
$$\hat{H} = \sum_i \left\{ E_J \left[1 - \cos\left(\varphi_{i+1} - \varphi_i\right) \right] + E_c \frac{q_i^2}{2} \right\}$$

Non-ergodic classical and quantum dynamics
Small entropy at infinite temperature.

M. G. Pino, L.B. Ioffe, BA

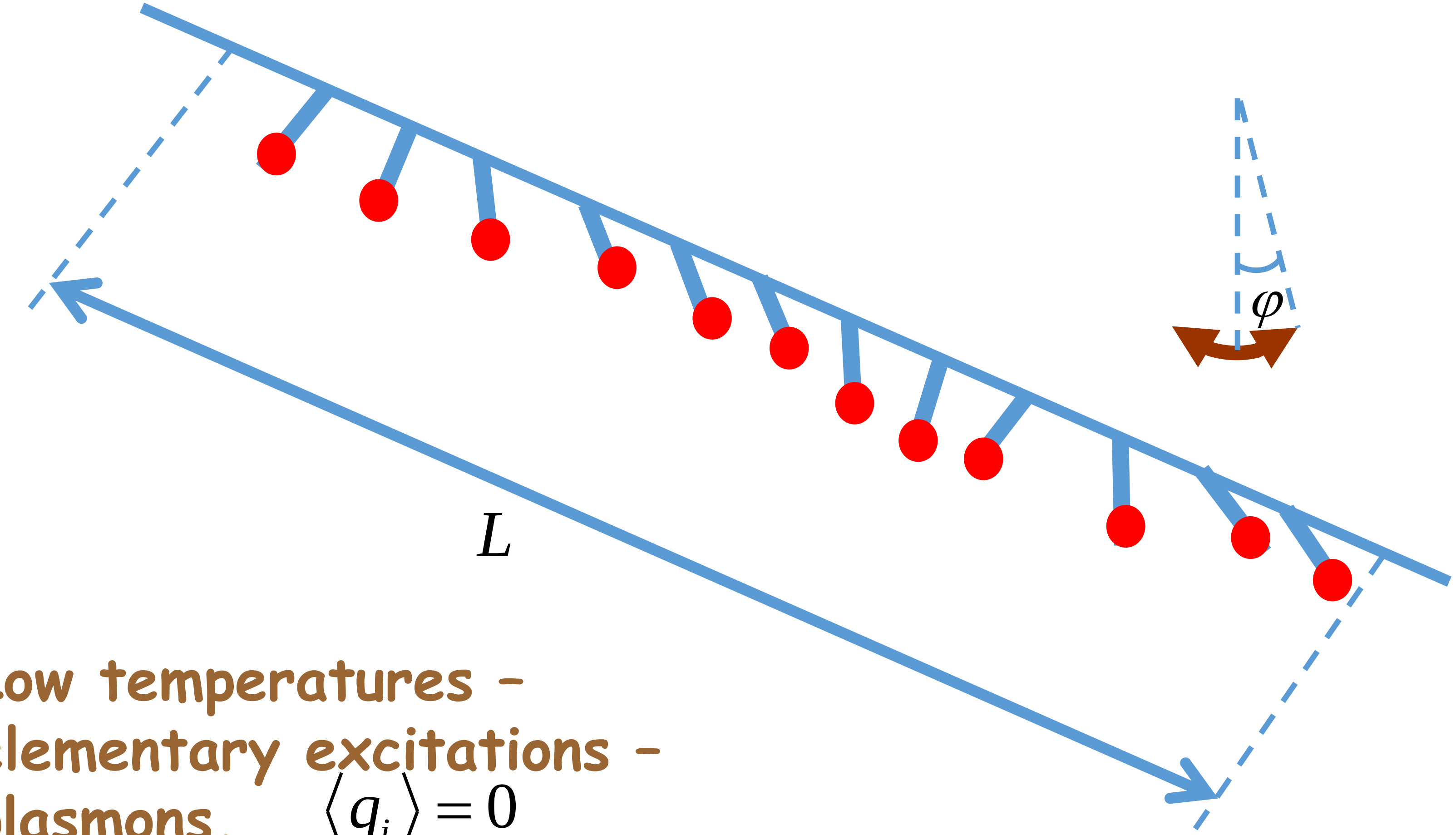
Ideal (no disorder) 1D Josephson array

$$\hat{H} = \sum_i \left\{ E_J \left[1 - \cos\left(\varphi_{i+1} - \varphi_i \right) \right] + E_c \frac{q_i^2}{2} \right\}$$



charge $q_i = \partial \varphi_i / \partial t$

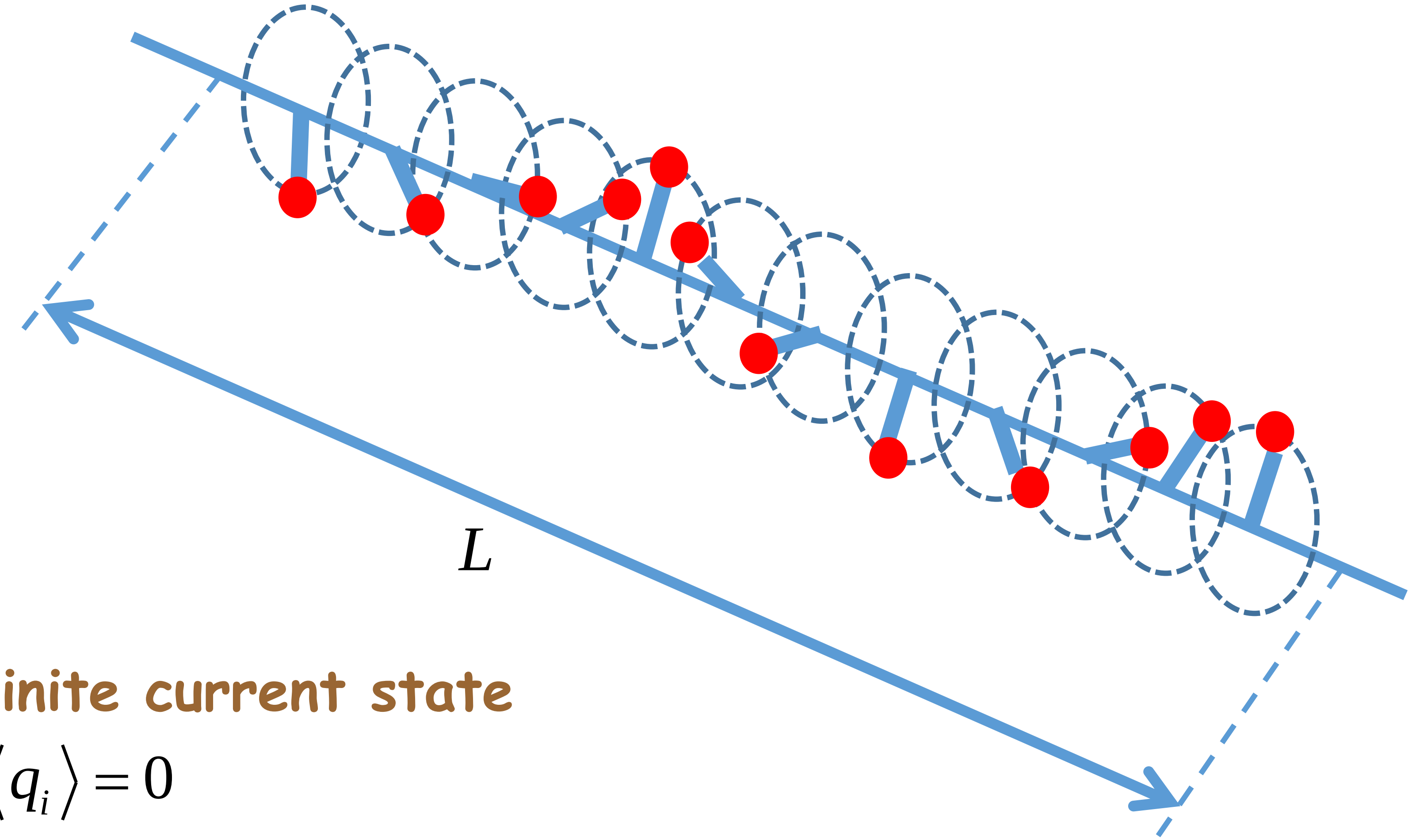
current $j_{i,i+1} = E_J \sin\left(\varphi_{i+1} - \varphi_i\right)$



Low temperatures –
elementary excitations –
plasmons. $\langle q_i \rangle = 0$

charge $q_i = \partial \varphi_i / \partial t$

current $j_{i,i+1} = E_J \sin\left(\varphi_{i+1} - \varphi_i\right)$



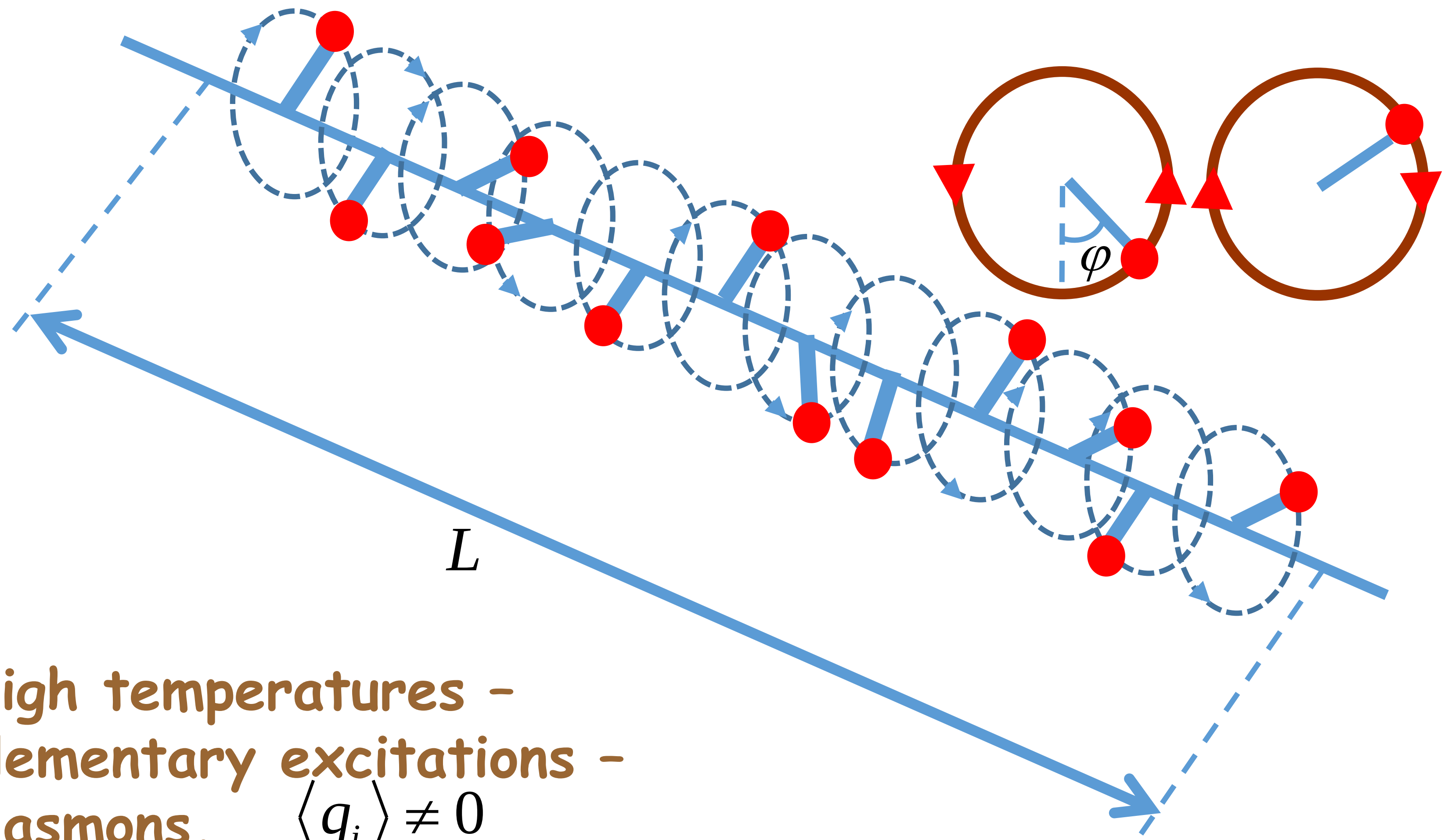
Finite current state

$$\langle q_i \rangle = 0$$

$$\langle j \rangle \neq 0$$

charge $q_i = \partial\varphi_i / \partial t$

current $j_{i,i+1} = E_J \sin(\varphi_{i+1} - \varphi_i)$



High temperatures – elementary excitations – plasmons. $\langle q_i \rangle \neq 0$

It makes more sense to build the description in terms of the charges rather than in terms of the phases.

Quantum Transition:

$$\hat{H} = \sum_i \left\{ E_J \left[1 - \cos\left(\varphi_{i+1} - \varphi_i\right) \right] + E_c \frac{q_i^2}{2} \right\} = \sum_i \left\{ \frac{E_J}{2} \left[\hat{b}_i^\dagger \hat{b}_{i+1} + \hat{b}_i \hat{b}_{i+1}^\dagger \right] + \frac{E_c}{2} q_i^2 \right\}$$

Matrix element of the $\left(q_i, q_{i+1}\right) \Rightarrow \left(q_i + 1, q_{i+1} - 1\right)$ transition is $E_J/2$

Energy difference of the two states $E_c\left(q_i - q_{i+1} + 1\right)$

$E_c q_i^2 \sim T \;\Rightarrow\; |q_i| \sim \sqrt{\dfrac{T}{E_c}}$ ⇨ Ratio $\sim \sqrt{\dfrac{T_c}{T}}$ $\quad T_c = \dfrac{E_J^2}{E_c}$

Therefore $T \ll T_c$ ⇨ metal
$T \gg T_c$ ⇨ insulator

Localized phase at *high* temperatures!

Freezing with cooling!

Classical limit: $\dfrac{E_J}{E_c} \to \infty$ ⇨ $T_c \to \infty$

Classical limit:

equations of motion:

$$\frac{\partial^2 \varphi_i}{\partial \tau^2} = \sin\left(\varphi_{i+1} - \varphi_i\right) + \sin\left(\varphi_{i-1} - \varphi_i\right)$$

$$\tau \equiv t\sqrt{E_J E_c}$$

$$T \leftarrow u \equiv \frac{U}{L} E_J^{-1}$$

U Total energy

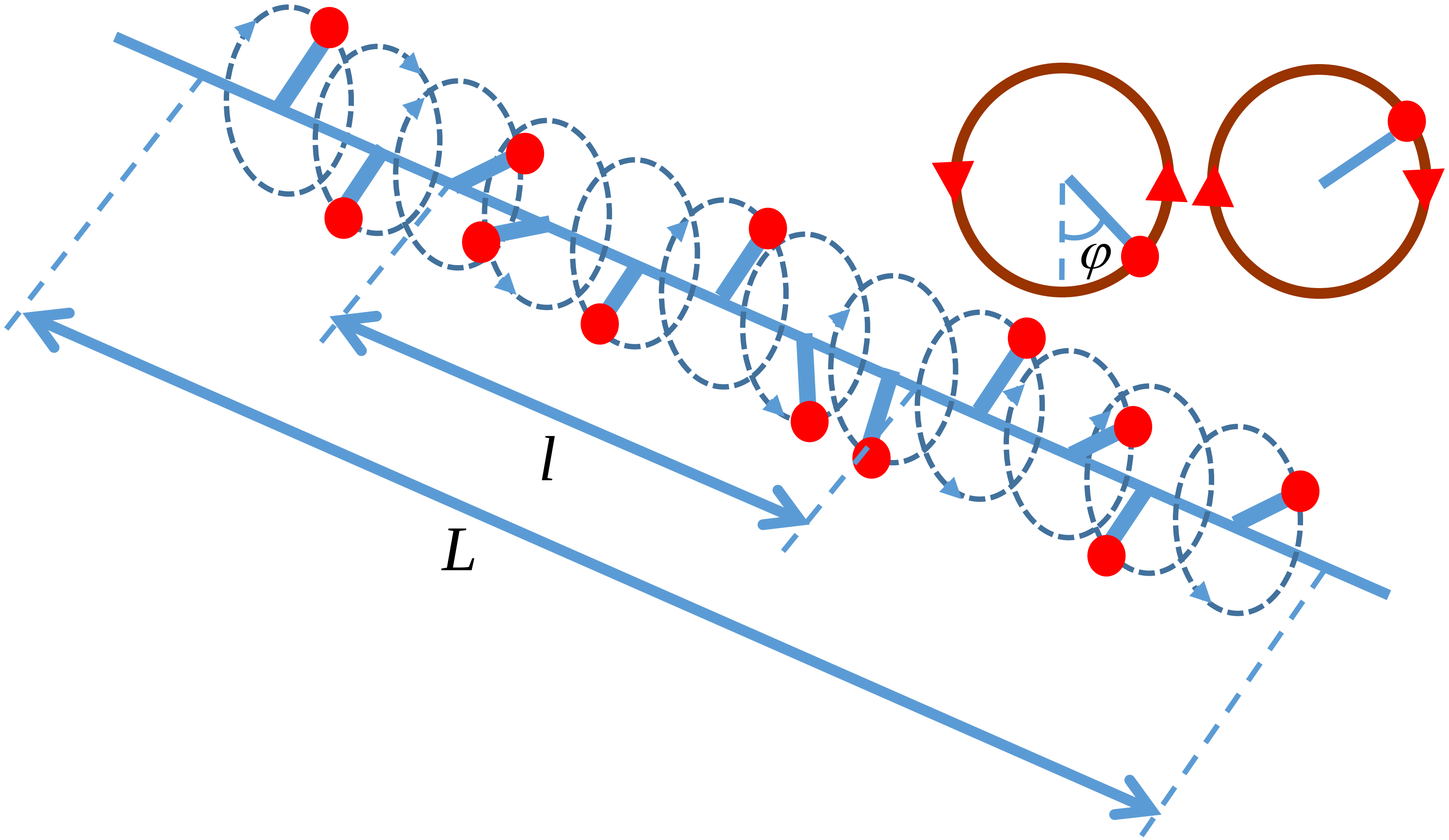
L Length = # of islands

$$u = \frac{1}{L}\sum_i \left\{ \frac{1}{2}\left(\frac{\partial \varphi_i}{\partial \tau}\right)^2 - \cos\left(\varphi_i - \varphi_{i-1}\right) \right\}$$

Dimensionless energy per island.

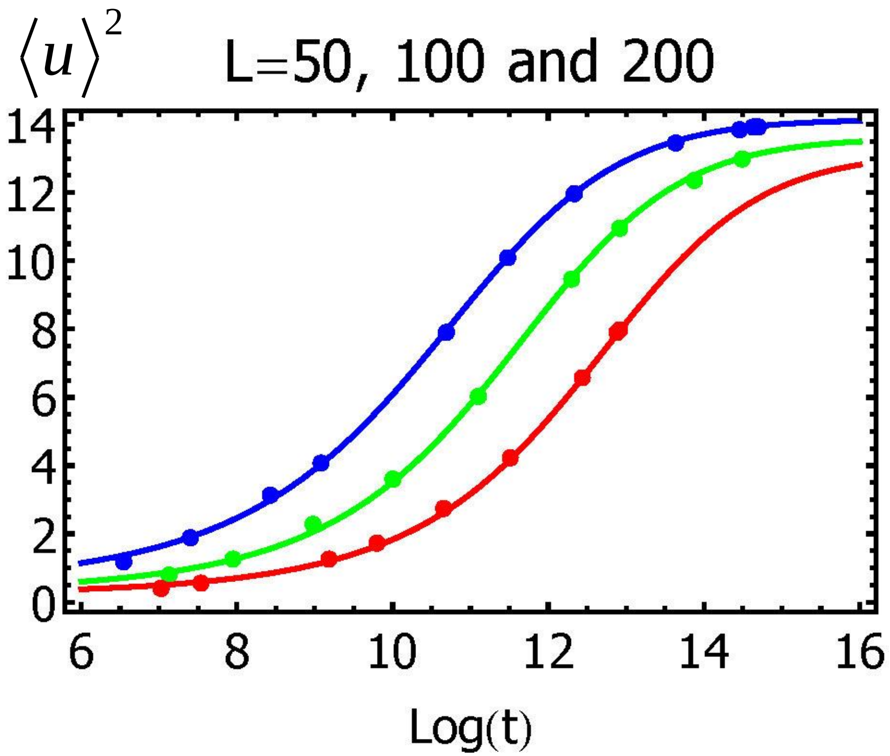
charge $q_i = \partial \varphi_i / \partial t$

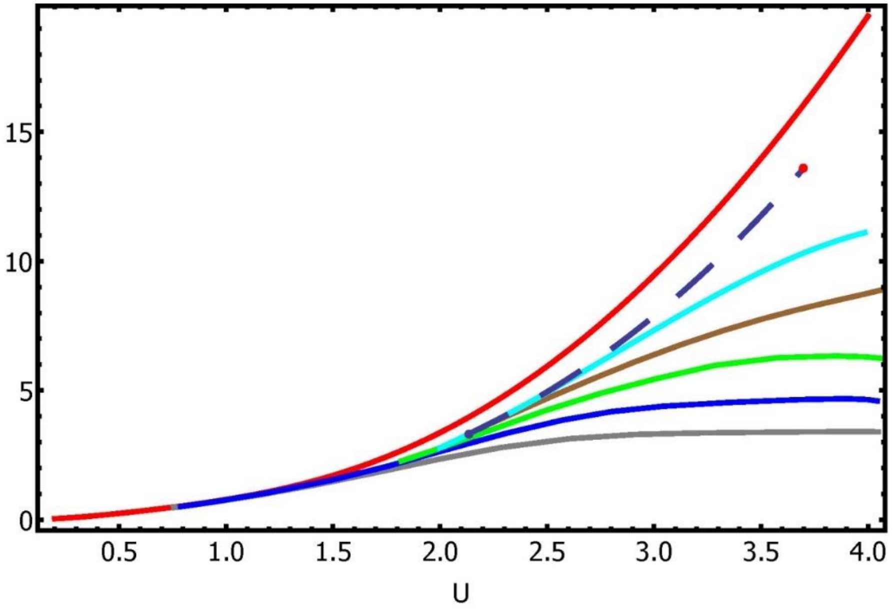
current $j_{i,i+1} = E_J \sin(\varphi_{i+1} - \varphi_i)$



Averaging over the "macroscopic" subsystem of the length $l \ll L$

Slow relaxation in the classical limit





$$\langle u^2 \rangle_\tau = \frac{\langle u^2 \rangle_\infty}{1 + \beta \exp\left[-\alpha \ln^2(\tau/\tau_0)\right]}$$

$$\langle U^2 \rangle = \frac{dU}{d\beta} \qquad U = F - T\frac{dF}{dT}$$

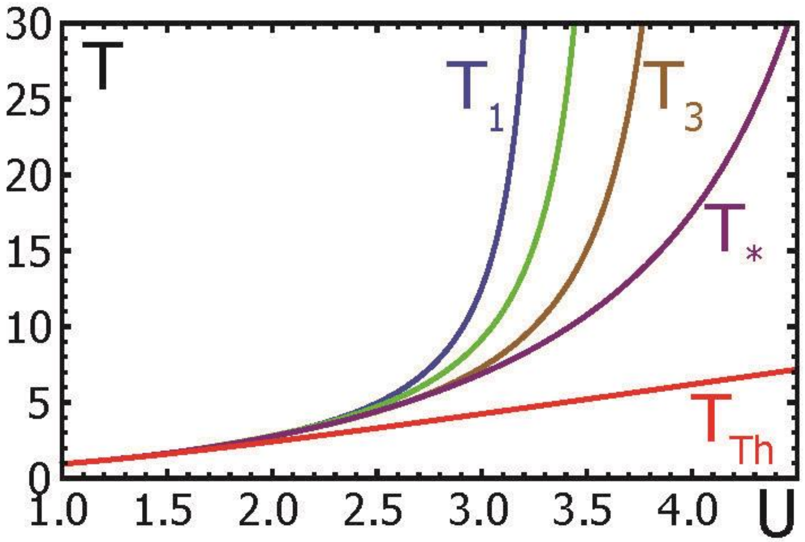
$$F = -TL\left[\frac{1}{2}\ln T + \ln J_0(1/T)\right]$$

Red curve

Averaging over the "macroscopic" subsystem of the length $l \ll L$

$$\langle u^2 \rangle_\infty \neq -T^2 \frac{du}{dT}$$!

Effective temperature



Effective temperature grows as ***u*** increases, but is different from both thermodynamic and pseudo-thermodynamic temperature. For example at ***u***=3.5:

$$T_{FDT} \approx 9.0 - 10$$

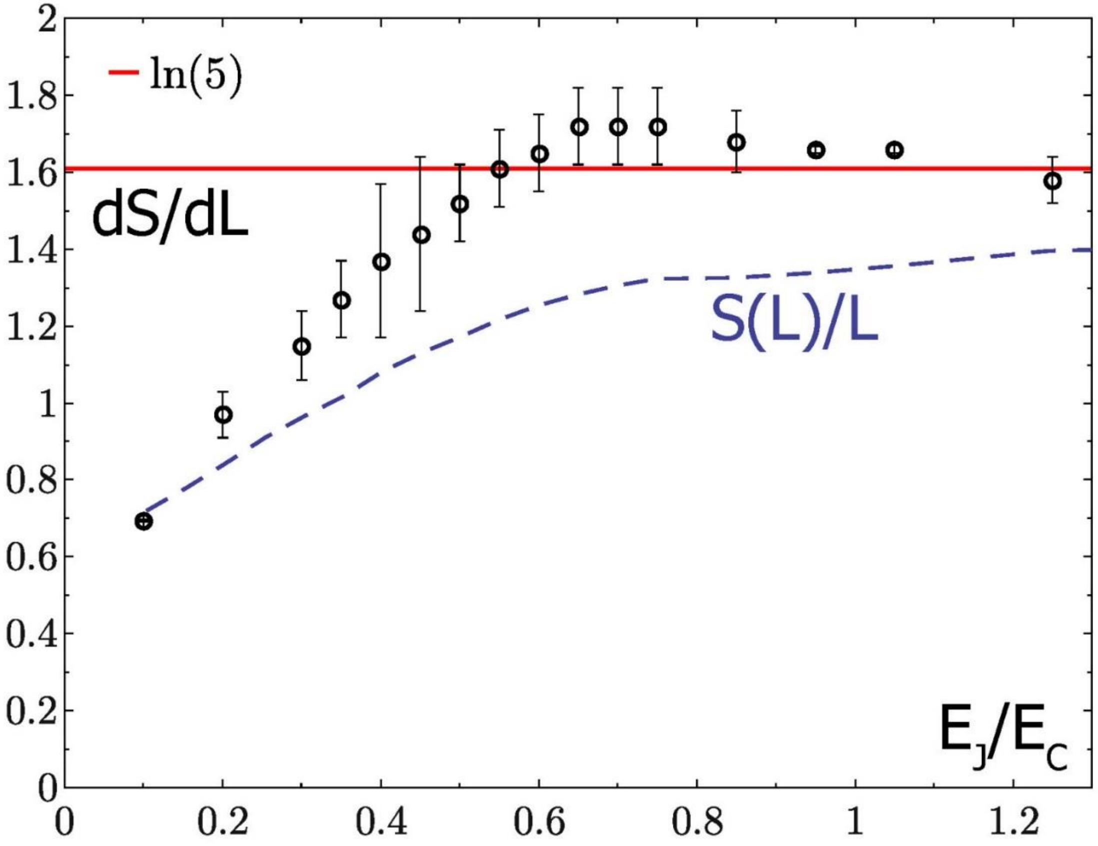
$$T_* \approx 10 - 11.0$$

$$T_{Th} \approx 5.3$$

Quantum Simulations

Finite number (5) of the charged states per site

$$q = 0, \pm 1, \pm 2$$



Entropy around critical point

$$T = \infty$$

Entanglement entropy coincides with the conventional entropy

Quantum Simulations

Finite number (5) of the charged states per site

$$q = 0, \pm 1, \pm 2$$