RENORMALIZATION GROUP: AN INTRODUCTION

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The renormalization group has played a crucial role in 20th century physics in two apparently unrelated domains: the theory of fundamental interactions at the microscopic scale and the theory of continuous macroscopic phase transitions. In the former framework, it emerged as a consequence of the necessity of renormalization to cancel infinities that appear in a straightforward interpretation of quantum field theory, and of the freedom of then defining the parameters of the renormalized theory at different momentum scales.

In the statistical physics of phase transitions, a more general renormalization group, based on a recursive averaging over short distance degrees of freedom, was later introduced to explain the universal properties of continuous phase transitions.

The renormalization group of quantum field theory now is understood as the asymptotic form of the general renormalization group in some neighbourhood of the Gaussian fixed point.

Therefore, in the framework of statistical field theories relevant for simple phase transitions, we explain first the perturbative renormalization group. We then review a few important applications like the proof of scaling laws and the determination of singularities of thermodynamic functions at the transition.

We then generalize the results to critical dynamics.

Finally, we describe the general renormalization group also called functional or exact renormalization group.

For an elementary introduction to the renormalization group, *cf.*, for example,

J. Zinn-Justin, *Phase transitions and renormalization group*, Oxford Univ. Press (Oxford 2007),

initially published in French *Transitions de phase et groupe de renormalisation.* EDP Sciences/CNRS Editions, Les Ulis 2005,

including the functional renormalization group in Chapter 16.

In www.scholarpedia.org, see

Jean Zinn-Justin (2010), *Critical Phenomena: field theoretical approach*, Scholarpedia, 5(5):8346.

More advanced material can be found in

J. Zinn-Justin, *Quantum Field Theory and Critical Phenomena*, Clarendon Press 1989 (Oxford 4th ed. 2002),

including critical dynamics in Chapter 36.

Statistical field theory

In the theory of continuous phase transitions, one is interested in the large distance behaviour or macroscopic properties of physical observables near the transition temperature $T = T_c$. At the critical temperature, the correlation length, which defines the scale on which correlations above T_c decay exponentially, diverges and correlation functions decay only algebraically. This gives rise to non-trivial large distance properties that are, to a large extent, independent of the short distance structure, a property called universality.

Intuitive arguments indicate that even if the initial statistical model is defined in terms of random variables associated to the sites of a space lattice, and which take only a finite set of values (like, *e.g.*, the classical spins of the Ising model), when the correlation length is large, the large distance behaviour can be inferred from a statistical field theory in continuum space.

Therefore, we consider a classical statistical system defined in terms of a random real field $\phi(x)$ in continuum space, $x \in \mathbb{R}^d$, and a functional measure on fields of the form $\mathrm{e}^{-\mathcal{H}(\phi)}/\mathcal{Z}$, where $\mathcal{H}(\phi)$ is called the Hamiltonian in statistical physics and $\mathcal{Z}$ is the partition function (a normalization) given by the field integral (*i.e.*, a sum over field configurations)

$$\mathcal{Z} = \int [\mathrm{d}\phi(x)]\, \mathrm{e}^{-\mathcal{H}(\phi)},$$

where the dependence in the temperature T is included in $\mathcal{H}(\phi)$.

The essential condition of short range interactions in the initial statistical system translates into the property of locality of the field theory: $\mathcal{H}(\phi)$ can be chosen as a space-integral over a linear combination of monomials in the field and its derivatives.

We assume also space translation and rotation invariance and, to discuss a specific case, $\mathbb{Z}_2$ reflection symmetry (like in the Ising model): $\mathcal{H}(\phi)=\mathcal{H}(-\phi)$. In d space dimensions, a typical form then is

$$\mathcal{H}(\phi) = \int \mathrm{d}^d x \left[\tfrac{1}{2}\left(\nabla_x \phi(x)\right)^2 + \tfrac{1}{2} r \phi^2(x) + \frac{g}{4!}\phi^4(x) + \cdots\right].$$

In the low temperature phase $T < T_c$, the $\mathbb{Z}_2$ symmetry is spontaneously broken.

Finally, as a systematic expansion of corrections to the mean field approximation indicates, the coefficients of $\mathcal{H}(\phi)$, like above $r, g\ldots$, are regular functions of the temperature T near the critical temperature T_c.

Correlation functions

Physical observables involve field correlation functions (generalized moments of the field distribution),

$$\langle \phi(x_1)\phi(x_2)\ldots\phi(x_n)\rangle \equiv \frac{1}{\mathcal{Z}}\int [\mathrm{d}\phi(x)]\phi(x_1)\phi(x_2)\ldots\phi(x_n)\,\mathrm{e}^{-\mathcal{H}(\phi)}.$$

They can be derived by functional differentiation from the generating functional (generalized partition function) in an external field $H(x)$,

$$\mathcal{Z}(H) = \int [\mathrm{d}\phi(x)]\,\exp\left[-\mathcal{H}(\phi) + \int \mathrm{d}^d x\, H(x)\phi(x)\right],$$

as

$$\langle \phi(x_1)\phi(x_2)\ldots\phi(x_n)\rangle = \frac{1}{\mathcal{Z}(0)}\,\frac{\delta}{\delta H(x_1)}\frac{\delta}{\delta H(x_2)}\cdots\frac{\delta}{\delta H(x_n)}\mathcal{Z}(H)\bigg|_{H=0}\,.$$

Connected correlation functions

More relevant physical observables are the connected correlation functions (generalized cumulants). The n-point function $W^{(n)}(x_1, x_2, \ldots, x_n)$ can be derived by functional differentiation from the free energy $\mathcal{W}(H) = \ln \mathcal{Z}(H)$:

$$W^{(n)}(x_1, x_2, \ldots, x_n) = \frac{\delta}{\delta H(x_1)} \frac{\delta}{\delta H(x_2)} \cdots \frac{\delta}{\delta H(x_n)} \mathcal{W}(H) \bigg|_{H=0} .$$

Translation invariance then implies

$$W^{(n)}(x_1, x_2, \ldots, x_n) = W^{(n)}(x_1 + a, x_2 + a, \ldots, x_n + a) \quad \forall a .$$

Connected correlation functions have the so-called cluster property: if in a connected n-point function one separates the points $x_1, \ldots, x_n$ into two non-empty sets, the function vanish when the distance between the two sets goes to infinity. It is the large distance behaviour of connected correlation functions in the critical domain near T_c that may exhibit universal properties.

Taking into account translation invariance, one also defines the Fourier transforms

$$(2\pi)^d \delta^{(d)} \left(\sum_{i=1}^{n} p_i \right) \tilde{W}^{(n)}(p_1, \dots, p_n)$$

$$= \int \mathrm{d}^d x_1 \dots \mathrm{d}^d x_n \, W^{(n)}(x_1, \dots, x_n) \exp\left(i \sum_{j=1}^{n} x_j p_j \right),$$

where, in analogy with quantum mechanics, the Fourier variables p_i are called momenta.

Finally, one introduces a generalized thermodynamic potential $\Gamma(\phi)$, Legendre transform of $\mathcal{W}(H)$ (like Hamiltonian and Lagrangian in classical mechanics). Its expansion in powers of ϕ defines vertex functions $\Gamma^{(n)}$:

$$\Gamma(\phi) = \sum_{n} \frac{1}{n!} \int \mathrm{d}^d x_1 \dots \mathrm{d}^d x_n \, \phi(x_1) \dots \phi(x_n) \Gamma^{(n)}(x_1, \dots, x_n).$$

Quadratic Hamiltonians and Gaussian measures

In the spirit of the central limit theorem of probabilities, one could expect that the universal properties of phase transitions can be described by Gaussian or weakly perturbed Gaussian measures, since they result from an averaging over many degrees of freedom.

To a Gaussian measure corresponds a quadratic Hamiltonian. The simplest form satisfying all conditions in d space dimensions (we assume $d \geq 2$), is ($\alpha_0 \geq 0$ constant)

$$\mathcal{H}^{(0)}(\phi) = \int \mathrm{d}^d x \left[\tfrac{1}{2} \bigl(\nabla_x \phi(x) \bigr)^2 + \tfrac{1}{2} \alpha_0 \phi^2(x) \right]. \tag{1}$$

One immediately verifies that the Gaussian model can describe only the high temperature phase $T \geq T_c$.

In the case of a Gaussian measure, all correlation functions can be expressed in terms of the two-point function with the help of Wick's theorem.

The Gaussian two-point function

The two-point function is the only connected correlation function. It has the Fourier representation (setting $\alpha_0 = m^2$)

$$W^{(2)}(x,0) = \frac{1}{(2\pi)^d} \int \frac{\mathrm{d}^d p \, \mathrm{e}^{ipx}}{m^2 + p^2}.$$

At large distance $|x| \to \infty$, for $m \neq 0$, correlations decrease exponentially as

$$W^{(2)}(x,0) \propto \frac{1}{|x|^{(d-1)/2}} \, \mathrm{e}^{-|x|/\xi},$$

where $\xi = 1/m$ is the correlation length.

At the critical point ($T = T_c$), the correlation length diverges, which implies $m = \alpha_0 = 0$ and, for $d > 2$, one finds the algebraic critical behaviour

$$W^{(2)}(x,0) = \frac{\Gamma(d/2 - 1)}{4\pi^{d/2}} \frac{1}{|x|^{d-2}} \, .$$

For $d = 2$, the Gaussian model is not defined at T_c.

Weakly perturbed or quasi-Gaussian model

To describe physics in the ordered phase below T_c, one needs to perturbe the quadratic Hamiltonian by adding higher power terms to the quadratic potential. Near the transition, the expectation value of the field is small and thus we can make a small field expansion:

$$\mathcal{H}(\phi) = \mathcal{H}^{(0)}(\phi) + \frac{g}{4!}\int \mathrm{d}^d x\, \phi^4(x) + \cdots \ .$$

Thermodynamic quantities can then be calculated by expanding in powers of the perturbation.

However, the form of the Hamiltonian $\mathcal{H}^{(0)}$ leads to a first unphysical problem: too singular, not even continuous, fields contribute to the field integral in such a way that correlation functions at coinciding points are not defined. For example,

$\langle \phi^2(x) \rangle = W^{(2)}(0,0) = \dfrac{1}{(2\pi)^d}\displaystyle\int \frac{\mathrm{d}^d p}{p^2+\alpha_0}$, diverges in all space dimensions $d \geq 2$.

Regularization

This problem was absent in the initial statistical model, due to its short distance structure. Thus, it is necessary to introduce an effective short distance structure, that is, to modify the Gaussian measure to restrict the field integration to more regular fields, continuous to define expectation values of powers of the field at the same point, satisfying differentiability conditions to define expectation values of the field and its derivatives taken at the same point, a procedure called regularization.

This can be achieved by adding to $\mathcal{H}^{(0)}(\phi)$ enough terms with more derivatives

$$\mathcal{H}^{(0)}(\phi) \mapsto \mathcal{H}_{\mathrm{G}}(\phi) = \mathcal{H}^{(0)}(\phi) + \frac{1}{2} \sum_{k=2}^{2k_{\max}} \alpha_k \int \mathrm{d}^d x\, \phi(x) \nabla_x^{2k} \phi(x).$$

For example, simple continuity requires $2k_{\max} > d$.

After Fourier transformation, this modification has the effect of suppressing the contribution of field components corresponding to momenta $|p| \gg 1$.

Perturbative expansion at criticality and the role of dimension four

An analysis of the expansion in powers of the ϕ^4 interaction, at criticality ($T = T_c$), reveals that the perturbative expansion is then well behaved for dimensions $d \geq 4$, though it is affected by large logarithms in $d = 4$. However, for $d < 4$ it is affected by small momentum, equivalently large distance divergences. Moreover, in a systematic small field expansion, the ϕ^4 interaction generates the most singular contributions and thus is singled out.

Since the dimensions of physical interest are $d \leq 4$, this is a very relevant issue. To deal with this very difficult problem, a completely new strategy has been invented, based of the renormalization group idea.

The renormalization group: General idea

To construct a RG flow in continuum space, the basic idea is to integrate in the field integral recursively over short distance degrees of freedom. This leads to the definition of an effective Hamiltonian $\mathcal{H}_\lambda$ function of a scale parameter $\lambda > 0$ (such that $\mathcal{H}_1 = \mathcal{H}$) and of a transformation $\mathcal{T}$ in the space of Hamiltonians such that

$$\lambda \frac{\mathrm{d}}{\mathrm{d}\lambda} \mathcal{H}_\lambda = \mathcal{T}\left[\mathcal{H}_\lambda\right], \tag{2}$$

a flow equation called RG equation (RGE). The appearance of the derivative $\lambda \mathrm{d}/\mathrm{d}\lambda = \mathrm{d}/\mathrm{d}\ln\lambda$ reflects the multiplicative character of dilatations. The RGE thus defines a dynamical process in the "time" $\ln\lambda$. The denomination renormalization group (RG) refers to the property that $\ln\lambda$ belongs to the additive group of real numbers.

RG equation: General structure, fixed points

We will construct a RG flow that defines a stationary Markov process, that is, $\mathcal{T}[\mathcal{H}_\lambda]$ depends on $\mathcal{H}_\lambda$ but not on the trajectory that has led from $\mathcal{H}_{\lambda=1}$ to $\mathcal{H}_\lambda$, and depends on λ only through $\mathcal{H}_\lambda$.

Universality is then related to the existence of fixed points, solution of

$$\mathcal{T}(\mathcal{H}^*) = 0\,.$$

We also assume that the mapping $\mathcal{T}$ is differentiable so that, near a fixed point, the RG flow can be linearized,

$$\mathcal{T}(\mathcal{H}^* + \Delta\mathcal{H}_\lambda) \sim L^* \Delta\mathcal{H}_\lambda\,,$$

and is governed by the eigenvalues and eigenvectors of the linear operator L^*. Formally, the local solution of the linearized equations can be written as

$$\mathcal{H}_\lambda = \mathcal{H}^* + \lambda^{L^*}\left(\mathcal{H}_{\lambda=1} - \mathcal{H}^*\right).$$

The Gaussian fixed point

A RG can be constructed that has the Gaussian model as a fixed point. The Hamiltonian flow can be implemented by the simple scaling

$$\phi(x) \mapsto \lambda^{(2-d)/2}\phi(x/\lambda). \tag{3}$$

After the change of variables $x' = x/\lambda$, one verifies that the Hamiltonian

$$\mathcal{H}^*_{\mathrm{G}}(\phi) = \frac{1}{2}\int \mathrm{d}^d x \bigl(\nabla_x \phi(x)\bigr)^2, \tag{4}$$

corresponding to the critical Gaussian model, is invariant. The RG has $\mathcal{H}^*_{\mathrm{G}}$ as a fixed point. The Hamiltonian flow (3) corresponds in fact to the linear approximation of the general RG near the Gaussian fixed point.

The linearized RG flow

The transformation (3) generates the linearized RG flow at the Gaussian fixed point. Eigenvectors of the linear flow (3) are monomials of the form

$$\mathcal{O}_{n,k}(\phi) = \int \mathrm{d}^d x \, O_{n,k}(\phi, x),$$

where $O_{n,k}(\phi, x)$ is a product of powers of the field and its derivatives at point x with $2n$ powers of the field (reflection $\mathbb{Z}_2$ symmetry) and $2k$ powers of ∂_μ.

Their RG behaviour under the transformation (3) is then given by a simple dimensional analysis. One defines the dimension of x as -1 and the (Gaussian) dimension of the field is $[\phi] = (d-2)/2$. The dimension $[\mathcal{O}_{n,k}]$ of $\mathcal{O}_{n,k}$ is then

$$[\mathcal{O}_{n,k}] = -d + n(d-2) + 2k \,. \tag{5}$$

It can be verified that $\mathcal{O}_{n,k}$ scales like $\lambda^{-[\mathcal{O}_{n,k}]}$, and the corresponding eigenvalue of L^* thus is $\ell_{n,k} = -[\mathcal{O}_{n,k}]$.

Discussion

When $\lambda \to +\infty$,

(i) for $\ell_{n,k} > 0$ the amplitude of $\mathcal{O}_{n,k}(\phi)$ increases; it is a direction of instability and in the RG terminology $\mathcal{O}_{n,k}(\phi)$ is a relevant perturbation;

(ii) for $\ell_{n,k} < 0$, the amplitude of $\mathcal{O}_{n,k}(\phi)$ decreases; it is a direction of stability and $\mathcal{O}_{n,k}(\phi)$ is an irrelevant perturbation;

(iii) in the special case $\ell_{n,k} = 0$, one speaks of a marginal perturbation and the linear approximation is no longer sufficient to discuss stability. Logarithmic behaviour in λ is then expected (we omit here unphysical redundant perturbations).

Since $\ell_{1,0} = 2$, $\int \mathrm{d}^d x\, \phi^2(x)$ corresponds always to a direction of instability: indeed it induces a deviation from the critical temperature and thus a finite correlation length.

For $d > 4$, no other perturbation is relevant and the Gaussian fixed point is stable on the critical surface ($\xi = \infty$).

Since $\ell_{2,0} = 4-d$, at $d = 4$ one perturbation, $\int \mathrm{d}^d x\, \phi^4(x)$, becomes marginal which below dimension four becomes relevant. In dimension $d = 4-\varepsilon$, $\varepsilon > 0$ small (a notion we define later), it is the only relevant perturbation and one expects to be able to describe critical properties with a Gaussian theory to which this unique term is added.

To summarize, for systems with a $\mathbb{Z}_2$ or, more generally, with an $O(N)$ symmetry, one concludes that

(i) the Gaussian fixed point is stable above space dimension four;

(ii) from a next order analysis, one shows that it is marginally stable in dimension four;

(iii) it is unstable below dimension four.

Rescaling and Gaussian renormalization

We now assume that the initial Hamiltonian is very close to the Hamiltonian of the Gaussian fixed point. The RG flow is then very close to the local linear flow. Therefore, we first perform the corresponding RG transformation. We introduce a parameter $\Lambda \gg 1$ and substitute

$$\phi(x) \mapsto \Lambda^{(2-d)/2}\phi(x/\Lambda).$$

In quantum field theory, this could be called a Gaussian renormalization. After the change of variables $x' = x/\Lambda$, a monomial $\mathcal{O}_{n,k}(\phi)$ is multiplied by $\Lambda^{-[\mathcal{O}_{n,k}]}$, where $[\mathcal{O}_{n,k}]$ is the dimension in the sense of the linearized RG. The Gaussian RG scaling can thus be inferred from the dimensions given by Λ: coordinates x have dimension Λ^{-1}, derivatives and momenta dimension Λ and the field dimension $\Lambda^{(d-2)/2}$. The Hamiltonian is dimensionless.

In the context of quantum field theory, since the regularization has then the effect, in the Fourier representation, to suppress field contributions with momenta $|p| \gg \Lambda$ in the perturbative expansion, Λ is also called the cut-off.

Statistical field theory: Perturbative expansion

The Gaussian model in the critical domain

After rescaling, the Hamiltonian of the Gaussian model takes the form

$$\mathcal{H}_{\mathrm{G}}(\phi) = \frac{1}{2}\int \mathrm{d}^d x \left[\left(\nabla_x \phi(x)\right)^2 + \alpha_0 \Lambda^2 \phi^2(x) + \sum_{k=2} \alpha_k \Lambda^{2-2k} \phi(x) \nabla_x^{2k} \phi(x) \right],$$

where α_0 is the amplitude of the only relevant term. For $\alpha_0 = 0$, except for the two-point function at coinciding points, one can take the $\Lambda \to \infty$ limit. However, for $\alpha_0 \neq 0$, to obtain a non-trivial universal large distance behaviour, it is also necessary to compensate the effect of the RG flow by choosing α_0 infinitesimal, that is, by taking the $\Lambda \to \infty$ limit at $r = \alpha_0 \Lambda^2$ fixed (a Gaussian mass renormalization in quantum field theory language). This defines the critical domain.

The weakly perturbed or quasi-Gaussian model

To allow for spontaneous $\mathbb{Z}_2$ symmetry breaking and, thus, to be able to describe physics below T_c, terms have necessarily to be added to the Gaussian Hamiltonian to generate a double-well potential for constant fields. The minimal addition, and the leading term from the RG viewpoint, is

$$\mathcal{H}_{\mathrm{G}} \mapsto \mathcal{H}(\phi) = \mathcal{H}_{\mathrm{G}}(\phi) + \frac{g}{4!}\Lambda^{4-d} \int \mathrm{d}^d x\, \phi^4(x), \quad g \geq 0\,.$$

The ϕ^4 term generates a shift of the critical temperature. To recover a critical theory ($T = T_c$), it is necessary to adjust the coefficient of the ϕ^2 term: $\alpha_0 = (\alpha_0)_c(g)$, a mass renormalization in the quantum field theory terminology, and this defines the critical Hamiltonian $\mathcal{H}_c$.

For $d > 4$, as we have shown the ϕ^4 term is then an irrelevant perturbation, as the power of Λ^{4-d} indicates, which does not invalidate the universal predictions of the Gaussian model.

Leading corrections to the Gaussian model are obtained by expanding in powers of the coefficient g of the ϕ^4 term.

In terms of $u = g\Lambda^{4-d}$, the partition function, for example, is given by

$$\mathcal{Z} = \sum_{k=0}^{\infty} \frac{(-u)^k}{(4!)^k k!} \left\langle \left(\int \mathrm{d}^d x \, \phi^4(x) \right)^k \right\rangle_{\mathrm{G}} .$$

The Gaussian expectations values $\langle \bullet \rangle_{\mathrm{G}}$ can then be evaluated in terms of the Gaussian two-point function with the help of Wick's theorem (Feynman graph expansion).

By contrast, for any $d < 4$, the ϕ^4 contribution is relevant: the Gaussian fixed point is unstable and no longer governs the large distance behaviour. The perturbative expansion of the critical theory ($T = T_c$) in powers of u contains so-called infra-red, that is, long distance, or zero momentum in the Fourier representation, divergences.

Renormalization group in dimension $d = 4 - \varepsilon$

For $d < 4$ fixed, the determination of the large distance behaviour of correlation functions requires the construction of a general renormalization group: this leads to functional equations that we describe later, but which, in general, unfortunately cannot be solved analytically.

However, a trick has been discovered to extend the definition of all terms of the perturbative expansion to arbitrary complex values of the dimension d in the form of meromorphic functions.

This allows replacing, in dimension $d = 4 - \varepsilon$ and in the framework of a double series expansion in g and ε, the general renormalization group by a much simpler asymptotic form and studying the model analytically. (Though a numerical method has been developed, based on the field theory RG in the form of Callan–Symanzik equations, that circumvents the problem of the ε-expansion but requires an additional assumption.)

Dimensional continuation and regularization

To discuss dimensional continuation, one introduces the Fourier representation of the two-point function (or propagator) $\Delta(x)$, corresponding to the Hamiltonian of the Gaussian model,

$$\Delta(x) \equiv \langle \phi(x)\phi(0) \rangle_{\mathrm{G}} = \frac{1}{(2\pi)^d} \int \mathrm{d}^d p \, \mathrm{e}^{-ipx} \tilde{\Delta}(p).$$

A representation of $\tilde{\Delta}(p)$ useful for dimensional continuation then is the Laplace representation (here written for the critical propagator)

$$\tilde{\Delta}(p) = \int_0^\infty \mathrm{d}s \, \rho(s\Lambda^2) \mathrm{e}^{-sp^2}, \tag{6}$$

where $\rho(s) \to 1$ when $s \to \infty$. Moreover, to reduce the field integration to continuous fields and, thus, to render the perturbative expansion finite, one needs at least $\rho(s) = O(s^q)$ with $q > (d-2)/2$ for $s \to 0$.

If, in addition, one wants the expectation values of all local polynomials to be defined, one must impose to $\rho(s)$ to converge to zero faster than any power for $s \to 0$.

A contribution to perturbation theory (represented graphically by a Feynman diagram) takes, in Fourier representation, the form of a product of propagators integrated over a subset of momenta. With the Laplace representation, all momentum integrations become Gaussian and can be performed, resulting in explicit analytic meromorphic functions of the dimension parameter d. This can be illustrated by two simple examples.

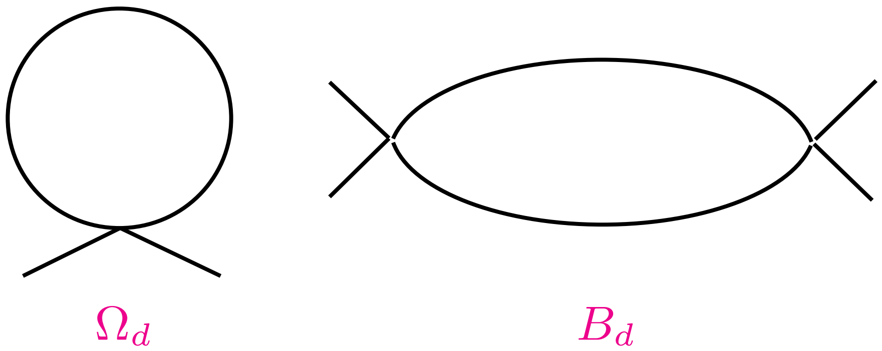

Fig. 1 Two one-loop diagrams.

The contribution of order g to the two-point function (figure 1) is proportional to

$$\Omega_d = \frac{1}{(2\pi)^d} \int \mathrm{d}^d k \, \tilde{\Delta}(k) = \frac{1}{(2\pi)^d} \int \mathrm{d}^d k \int_0^\infty \mathrm{d}s \, \rho(s\Lambda^2) \mathrm{e}^{-sk^2}$$
$$= \frac{1}{(4\pi)^{d/2}} \int_0^\infty \mathrm{d}s \, s^{-d/2} \rho(s\Lambda^2),$$

which, in the latter form, is holomorphic for $2 < \operatorname{Re} d < 2(1+q)$.

In the same way, the contribution of order g^2 to the four-point function (figure 1), is proportional to

$$\begin{aligned} B_d(p) &= \frac{1}{(2\pi)^d} \int \mathrm{d}^d k \, \tilde{\Delta}(k) \tilde{\Delta}(p-k) \\ &= \frac{1}{(2\pi)^d} \int \mathrm{d}^d k \int_0^\infty \mathrm{d}s_1 \, \mathrm{d}s_2 \, \rho(s_1\Lambda^2)\rho(s_2\Lambda^2) \mathrm{e}^{-s_1 k^2 - s_2 (p-k)^2} \\ &= \frac{1}{(4\pi)^{d/2}} \int_0^\infty \frac{\mathrm{d}s_1 \, \mathrm{d}s_2}{(s_1+s_2)^{d/2}} \rho(s_1\Lambda^2)\rho(s_2\Lambda^2) \, \mathrm{e}^{-p^2 s_1 s_2/(s_1+s_2)}, \end{aligned}$$

which, in the latter form, is holomorphic for $2 < \operatorname{Re} d < 4(1+q)$.

For the theory of critical phenomena, dimensional continuation is sufficient since it allows exploring the neighbourhood of dimension four, determining fixed points and calculating universal quantities as $\varepsilon = (4-d)$-expansions.

Dimensional regularization

However, for practical calculations, but then restricted to the leading large distance behaviour, an additional step is extremely useful. It can be verified that if one decreases $\operatorname{Re} d$ enough, so that by naive power counting all momentum integrals are convergent, one can, after explicit dimensional continuation, take the infinite Λ limit. The resulting perturbative contributions become meromorphic functions with poles at dimensions at which large momentum, and low momentum in the critical theory, divergences appear. This method of regularizing large momentum divergences is called dimensional regularization and is extensively used in quantum field theory. In the theory of critical phenomena, it has also been used to calculate universal quantities like critical exponents, as ε-expansions. For example,

$$B_d(p) = -\frac{2\pi\Gamma(d/2)}{(4\pi)^{d/2}\sin(\pi d/2)\Gamma(d-1)}p^{d-4} = \frac{1}{8\pi^2\varepsilon}(1-\varepsilon\ln p) + O(\varepsilon).$$

Perturbative renormalization group: the critical theory

The perturbative renormalization group, as it has been developed in the framework of the perturbative expansion of quantum field theory, relies on the so-called renormalization theory. For the ϕ^4 field theory it has been first formulated in space dimension $d = 4$. For critical phenomena, a minor extension is required that involves an additional expansion in powers of $\varepsilon = 4 - d$, after dimensional continuation.

We first consider the critical theory ($T = T_c$) corresponding to the Hamiltonian $\mathcal{H}_c(\phi)$.

To formulate the renormalization theorem, one introduces a momentum $\mu \ll \Lambda$, called the renormalization scale, and a parameter g_{r} characterizing the effective ϕ^4 coefficient at scale μ, called the renormalized coupling constant.

The renormalization theorem

One can then find two dimensionless functions $Z(\Lambda/\mu, g)$ and $Z_g(\Lambda/\mu, g)$ that satisfy (g and Λ/μ are the only two dimensionless combinations)

$$\Lambda^{4-d} g = \mu^{4-d} Z_g(\Lambda/\mu, g) g_\mathrm{r} = \mu^{4-d} g_\mathrm{r} + O(g^2),\ \ Z(\Lambda/\mu, g) = 1 + O(g),$$

calculable order by order in a double series expansion in powers of g and ε, such that all connected correlations functions

$$\tilde{W}_\mathrm{r}^{(n)}(p_i; g_\mathrm{r}, \mu, \Lambda) = Z^{-n/2}(g, \Lambda/\mu) \tilde{W}^{(n)}(p_i; g, \Lambda),$$

called renormalized, have, order by order in g_r, finite limits $\tilde{W}_\mathrm{r}^{(n)}(p_i; g_\mathrm{r}, \mu)$ when $\Lambda \to \infty$ at p_i, μ, g_r fixed.

The renormalization constant $Z^{1/2}(\Lambda/\mu, g)$ is the ratio between the full field renormalization in presence of the ϕ^4 interaction and the Gaussian field renormalization $\Lambda^{(d-2)/2}$.

Universality: a first step

There is some arbitrariness in the choice of the renormalization constants Z and Z_g since they can be multiplied by arbitrary functions of g_{r}. The renormalization constants can be completely determined by imposing two renormalization conditions to the renormalized correlation functions, which are then independent of the specific form of the short distance regularization.

This leads to a first very important result: since initial and renormalized correlation functions are proportional, they have the same large distance behaviour. This behaviour is thus to a large extent universal since it can depend at most on only one parameter, the ϕ^4 coefficient g.

Critical RG equations

From the relation between initial and renormalized functions and the existence of a limit $\Lambda \to \infty$, a new equation follows, obtained by differentiation of the equation with respect to Λ at μ, g_{r} fixed:

$$\Lambda \frac{\partial}{\partial \Lambda}\bigg|_{g_{\mathrm{r}},\mu \text{ fixed}} Z^{-n/2}(g, \Lambda/\mu)\tilde{W}^{(n)}(p_i; g, \Lambda) = \Lambda \frac{\partial}{\partial \Lambda}\bigg|_{g_{\mathrm{r}},\mu \text{ fixed}} \tilde{W}_{\mathrm{r}}^{(n)}(p_i; g_{\mathrm{r}}, \mu, \Lambda) \to 0\,.$$

In agreement with the perturbative philosophy, one then neglects all contributions that, order by order, decay as powers of Λ. One defines asymptotic functions $\tilde{W}_{\mathrm{as.}}^{(n)}(p_i; g, \Lambda)$ and $Z_{\mathrm{as.}}(g, \Lambda/\mu)$ as sums of the perturbative contributions to the functions $\tilde{W}^{(n)}(p_i; g, \Lambda)$ and $Z(g, \Lambda/\mu)$, respectively, that do not go to zero when $\Lambda \to \infty$. Then,

$$\Lambda \frac{\partial}{\partial \Lambda}\bigg|_{g_{\mathrm{r}},\mu \text{ fixed}} Z_{\mathrm{as.}}^{-n/2}(g, \Lambda/\mu)\tilde{W}_{\mathrm{as.}}^{(n)}(p_i; g, \Lambda) = 0\,.$$

Using the chain rule, one infers

$$\left[\Lambda\frac{\partial}{\partial\Lambda}+\beta(g,\Lambda/\mu)\frac{\partial}{\partial g}+\frac{n}{2}\eta(g,\Lambda/\mu)\right]\tilde{W}^{(n)}_{\text{as.}}(p_i;g,\Lambda)=0\,,$$

where the functions β and η are defined by

$$\beta(g,\Lambda/\mu)=\Lambda\frac{\partial}{\partial\Lambda}\bigg|_{g_{\text{r}},\mu} g\,,\quad \eta(g,\Lambda/\mu)=-\Lambda\frac{\partial}{\partial\Lambda}\bigg|_{g_{\text{r}},\mu}\ln Z_{\text{as.}}(g,\Lambda/\mu).$$

Since the functions $\tilde{W}^{(n)}_{\text{as.}}$ do not depend on μ, the functions β and η cannot depend on Λ/μ, and one finally obtains the RG equations (Zinn-Justin 1973):

$$\left(\Lambda\frac{\partial}{\partial\Lambda}+\beta(g)\frac{\partial}{\partial g}+\frac{n}{2}\eta(g)\right)\tilde{W}^{(n)}_{\text{as.}}(p_i;g,\Lambda)=0\,. \tag{7}$$

From the relation between g and g_{r}, one immediately infers that $\beta(g)=-\varepsilon g+O(g^2)$.

RG equations in the critical domain above T_c

Correlation functions may also exhibit universal properties near T_c when the correlation length ξ is large in the microscopic scale, here, $\xi\Lambda \gg 1$. To describe universal properties in the critical domain above T_c, one adds to the critical Hamiltonian the ϕ^2 relevant term:

$$\mathcal{H}_\tau(\phi) = \mathcal{H}_c(\phi) + \frac{\tau}{2}\int \mathrm{d}^d x\, \phi^2(x),$$

where $\tau \propto T - T_c \ll \Lambda^2$ characterizes the deviation from the critical temperature. The extended renormalization theorem involves a new renormalization factor $Z_2(\Lambda/\mu, g)$, ratio between the full renormalization of $\int \mathrm{d}^d x\, \phi^2(x)$ and the Gaussian renormalization. One then derives a more general RGE of the form (Zinn-Justin 1973)

$$\left[\Lambda\frac{\partial}{\partial\Lambda} + \beta(g)\frac{\partial}{\partial g} + \frac{n}{2}\eta(g) - \eta_2(g)\tau\frac{\partial}{\partial\tau}\right]\tilde{W}^{(n)}_{\text{as.}}(p_i;\tau,g,\Lambda) = 0\,,$$

where a new RG function $\eta_2(g)$, related to $Z_2(\Lambda/\mu, g)$, appears.

These equations can be further generalized to deal with an external field (a magnetic field for magnetic systems) and the corresponding induced field expectation value (magnetization for magnetic systems).

Renormalized RG equations

For $d = 4 - \varepsilon$, if one is only interested in the leading scaling behaviour (and the first correction), it is technically simpler to use dimensional regularization and the renormalized theory in the so-called minimal (or modified minimal) subtraction scheme. The relation between initial and renormalized correlation functions is asymptotically symmetric. One thus derives also (for the critical theory)

$$\left(\mu\frac{\partial}{\partial\mu} + \tilde{\beta}(g_{\mathrm{r}})\frac{\partial}{\partial g_{\mathrm{r}}} + \frac{n}{2}\tilde{\eta}(g_{\mathrm{r}})\right)\tilde{W}_{\mathrm{r}}^{(n)}(p_i, g_{\mathrm{r}}, \mu) = 0$$

with the definitions

$$\tilde{\beta}(g_{\mathrm{r}}) = \mu\frac{\partial}{\partial\mu}\bigg|_g g_{\mathrm{r}}\,, \quad \tilde{\eta}(g_{\mathrm{r}}) = \mu\frac{\partial}{\partial\mu}\bigg|_g \ln Z(g_{\mathrm{r}}, \varepsilon)\,.$$

In this scheme, the renormalization constants are obtained by continuation to low dimensions where the infinite Λ limit, at g_{r} fixed, can be taken. For example,

$$\lim_{\Lambda\to\infty} Z(\Lambda/\mu, g)\big|_{g_{\mathrm{r}}\ \text{fixed}} = Z(g_{\mathrm{r}}, \varepsilon).$$

Then, order by order in powers of g_{r}, they have a Laurent expansion in powers of ε. In the minimal subtraction scheme, the freedom in the choice of renormalization constants is used to reduce the Laurent expansion to the singular terms. For example, $Z(g_{\mathrm{r}}, \varepsilon)$ takes the form

$$Z(g_{\mathrm{r}}, \varepsilon) = 1 + \sum_{n=1}^{\infty} \frac{\sigma_n(g_{\mathrm{r}})}{\varepsilon^n} \text{ with } \sigma_n(g_{\mathrm{r}}) = O(g_{\mathrm{r}}^{n+1}).$$

A remarkable consequence is that the RG functions $\tilde{\eta}(g_{\mathrm{r}})$, and $\tilde{\eta}_2(g_{\mathrm{r}})$ when a ϕ^2 term is added, become independent of ε and $\tilde{\beta}(g_{\mathrm{r}})$ has the simple dependence $\tilde{\beta}(g_{\mathrm{r}}) = -\varepsilon g_{\mathrm{r}} + \tilde{\beta}_2(g_{\mathrm{r}})$, where $\tilde{\beta}_2(g_{\mathrm{r}}) = O(g_{\mathrm{r}}^2)$ is also independent of ε.

Solution of the RG equations: The epsilon-expansion

RG equations can be solved by the method of characteristics. In the simplest example of the critical theory, one introduces a scale parameter λ and two functions of $g(\lambda)$ and $\zeta(\lambda)$ defined by

$$\lambda\frac{\mathrm{d}}{\mathrm{d}\lambda}g(\lambda)=-\beta\bigl(g(\lambda)\bigr),\ g(1)=g\,,\quad \lambda\frac{\mathrm{d}}{\mathrm{d}\lambda}\ln\zeta(\lambda)=-\eta\bigl(g(\lambda)\bigr),\ \zeta(1)=1\,.$$

The function $g(\lambda)$ is the effective coefficient of the ϕ^4 term at the scale λ. One verifies that the differential RG equation then implies

$$\lambda\frac{\mathrm{d}}{\mathrm{d}\lambda}\left[\zeta^{n/2}(\lambda)\tilde{W}^{(n)}_{\text{as.}}\bigl(p_i;g(\lambda),\Lambda/\lambda\bigr)\right]=0\,,$$

from which one infers $(\Lambda\mapsto\lambda\Lambda)$

$$\tilde{W}^{(n)}_{\text{as.}}\bigl(p_i;g,\lambda\Lambda\bigr)=\zeta^{n/2}(\lambda)\tilde{W}^{(n)}_{\text{as.}}\bigl(p_i;g(\lambda),\Lambda\bigr).$$

From its definition, one verifies that $\tilde{W}^{(n)}_{\text{as.}}$ has dimension $(d-(d+2)n/2)$. Therefore,

$$\begin{aligned}\tilde{W}^{(n)}_{\text{as.}}(p_i/\lambda; g, \Lambda) &= \lambda^{(d+2)n/2-d}\tilde{W}^{(n)}_{\text{as.}}(p_i; g, \lambda\Lambda) \\ &= \lambda^{(d+2)n/2-d}\zeta^{n/2}(\lambda)\tilde{W}^{(n)}_{\text{as.}}(p_i; g(\lambda), \Lambda).\end{aligned}$$

These equations show that the general Hamiltonian flow reduces here to the flow of $g(\lambda)$ and, thus, the large distance behaviour is governed by the zeros of the function $\beta(g)$. Since $\beta(g) = -\varepsilon g + O(g^2)$, when $\lambda \to \infty$, if $g > 0$ is initially very small, it moves away from the unstable Gaussian fixed point, in agreement with the general RG analysis of the Gaussian fixed point.

If one assumes the existence of another zero $g^* > 0$ with then $\beta'(g^*) > 0$, then $g(\lambda)$ will converge toward this fixed point. Since $g(\lambda)$ tends toward the fixed point value g^*, and if $\eta(g^*) \equiv \eta$ is finite, one finds the universal behaviour

$$\tilde{W}^{(n)}_{\text{as.}}(p_i/\lambda; g, \Lambda) \underset{\lambda\to\infty}{\propto} \lambda^{(d+2-\eta)n/2-d} \tilde{W}^{(n)}_{\text{as.}}(p_i; g^*, \Lambda).$$

For the connected correlation functions in position space, this result translates into

$$W^{(n)}_{\text{as.}}(\lambda x_i; g, \Lambda) \underset{\lambda\to\infty}{\propto} \lambda^{-n(d-2+\eta)/2} W^{(n)}_{\text{as.}}(x_i; g^*, \Lambda),$$

for all x_i distinct.

The exponent $d_\phi = (d-2+\eta)/2$ is the dimension of the field ϕ, from the point of view of large distance properties.

Explicit calculations: the RG functions at one-loop

The inverse or vertex two-point function. At one-loop order,

$$\tilde{\Gamma}^{(2)}(p) = p^2 + r + \tfrac{1}{2} g \Omega_d + O(g^2),$$

where Ω_d is a constant given by the first diagram of figure 1. The critical theory is defined by

$$\tilde{\Gamma}^{(2)}(0) = 0$$

and this determines the critical value of the parameter r at order g:

$$r = r_c(g) \equiv -\tfrac{1}{2} g \Omega_d + O(g^2) \;\Rightarrow\; \tilde{\Gamma}^{(2)}(p) = p^2 + O(g^2).$$

Since β is of order g and $\tilde{\Gamma}^{(2)}$ satisfies the RG equation

$$\left(\Lambda \frac{\partial}{\partial \Lambda} + \beta(g) \frac{\partial}{\partial g} - \eta(g)\right) \tilde{\Gamma}^{(2)}(p; g, \Lambda) = 0 \;\Rightarrow\; \eta(g) = O(g^2)\,.$$

The four-point vertex (or 1PI) function. At one-loop order,

$$\tilde{\Gamma}^{(4)}(p_1, p_2, p_3, p_4) = \Lambda^{\varepsilon} g - \tfrac{1}{2} g^2 \Lambda^{2\varepsilon} \left[B_d(p_1 + p_2) + B_d(p_1 + p_3) \right. \\ \left. + B_d(p_1 + p_4) \right] + O(g^3),$$

where B_d is the second diagram of figure 1:

$$B_d(p) = \frac{1}{(2\pi)^d} \int \mathrm{d}^d q \, \tilde{\Delta}(q) \tilde{\Delta}(p - q) \underset{\Lambda \to \infty}{\sim} \frac{1}{(2\pi)^d} \int_{|q|<\Lambda} \frac{\mathrm{d}^d q}{q^4} \\ \underset{\Lambda \to \infty}{\sim} \frac{1}{8\pi^2} [\ln \Lambda + O(1)] + O(\varepsilon).$$

Thus, $\quad \tilde{\Gamma}^{(4)} = g + g\varepsilon \ln \Lambda - \dfrac{3g^2}{16\pi^2} \ln \Lambda + O(g^2) \times 1 + O(g^3, g^2 \varepsilon).$

The four-point satisfies

$$\left(\Lambda \frac{\partial}{\partial \Lambda} + \beta(g) \frac{\partial}{\partial g} - 2\eta(g) \right) \tilde{\Gamma}^{(4)}(p_i; g, \Lambda) = 0 \;\Rightarrow\; \beta = -\varepsilon g + \frac{3}{16\pi^2} g^2 + \cdots \;.$$

The RG β-function and the IR fixed point. Using the perturbative calculation of the two- and four-point functions at one-loop order, one has thus derived

$$\beta(g) = -\varepsilon g + \frac{3}{16\pi^2} g^2 + O(g^3, \varepsilon g^2).$$

In the sense of an ε-expansion, $\beta(g)$ has a zero g^* with a positive slope (Wilson–Fisher 1972)

$$g^* = \frac{16\pi^2\varepsilon}{3} + O(\varepsilon^2), \quad \omega = \beta'(g^*) = \varepsilon + O(\varepsilon^2),$$

which governs the large momentum behaviour of correlation functions. In addition, the exponent ω governs the leading correction to the critical behaviour.

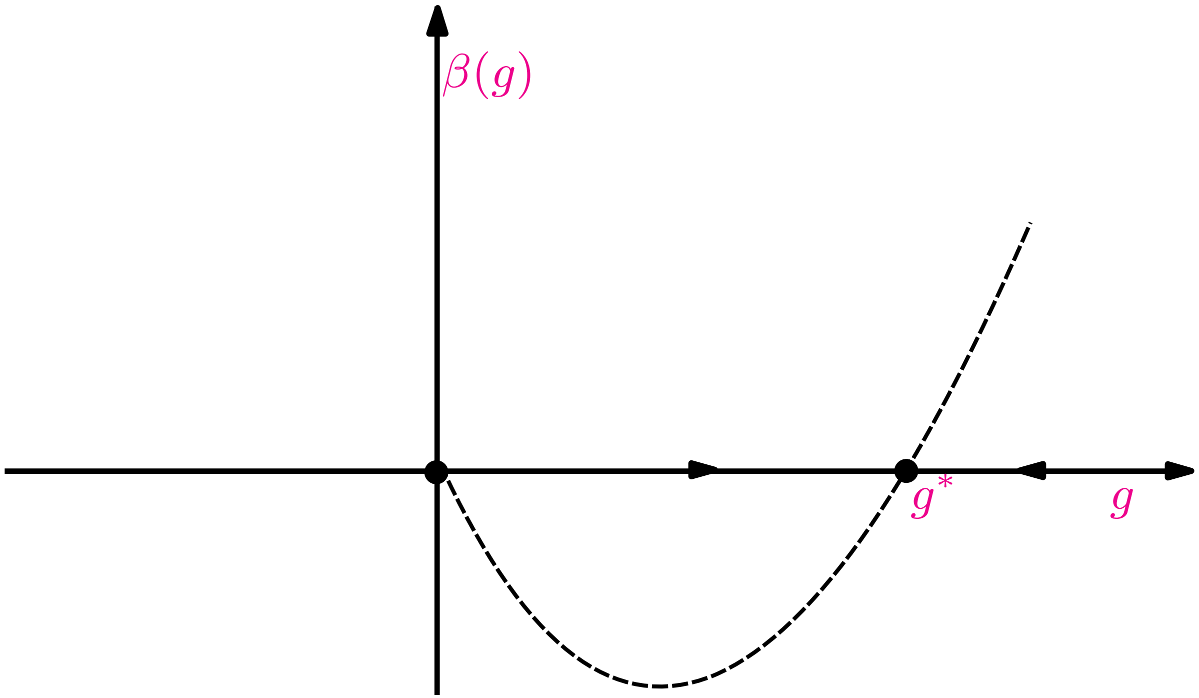

Fig. 2 The RG β-function and RG flow in the $(\phi^2)^2$ field theory for $d < 4$.

Generalization

The results obtained for models with a $\mathbb{Z}_2$ reflection symmetry can easily be generalized to N-vector models with $O(N)$ orthogonal symmetry, which belong to different universality classes.

Their universal properties can then be derived from an $O(N)$ symmetric field theory with an N-component field $\boldsymbol{\phi}(x)$ and a $g(\boldsymbol{\phi}^2)^2$ quartic term. Further generalizations involve theories with N-component fields but smaller symmetry groups, such that several independent quartic ϕ^4 terms are allowed. The structure of fixed points may then be more complicate.

Finally, correlation functions of the $O(N)$ model can be evaluated in the large N limit explicitly and the predictions of the ε-expansion can then be verified in this limit.

Epsilon-expansion: A few results

From the simple existence of the fixed point and of the corresponding ε-expansion, universal properties of an important class of critical phenomena can be proved to all orders in ε: this includes relations between critical exponents, scaling behaviour of correlation functions or the equation of state. Moreover, universal quantities can then be calculated as ε-expansions.

The scaling equation of state

The scaling properties of the equation of state of magnetic systems, that is, the relation between applied magnetic field H, magnetization M and temperature T, provide an example of the general results that can be obtained. In the relevant limit $|H| \ll 1$, $|T - T_c| \ll 1$, using RG arguments, one has proved Widom's conjectured scaling form

$$H = M^\delta f\big((T - T_c)/M^{1/\beta}\big),$$

where $f(z)$ is a universal function (up to normalizations).

Moreover, the exponents satisfy the relations

$$\delta = \frac{d+2-\eta}{d-2+\eta}, \quad \beta = \tfrac{1}{2}\nu(d-2+\eta),$$

where ν, the correlation length exponent, given by $\nu = 1/\big(\eta_2(g^*)+2\big)$, characterizes the divergence ξ of the correlation length at T_c:

$$\xi \propto |T-T_c|^{-\nu}.$$

Other relations can be derived, involving the magnetic susceptibility exponent γ characterizing the divergence of the two-point correlation function at zero momentum at T_c, or the exponent α characterizing the behaviour of the specific heat:

$$\gamma = \nu(2-\eta), \quad \alpha = 2-\nu d\,.$$

Note the relations involving the dimension d explicitly are not valid for the Gaussian fixed point.

Critical exponents as ε-expansions

As an illustration, we give here two successive terms of the ε-expansion of the exponents η, γ and ω for the $O(N)$ models, although the RG functions of the $(\phi^2)^2$ field theory are known to five-loop order and, thus, critical exponents are known up to order ε^5. In terms of the variable $v = N_d g$ where N_d is the loop factor

$$N_d = 2/(4\pi)^{d/2}\Gamma(d/2),$$

the RG functions $\beta(v)$ and $\eta_2(v)$ at two-loop order, $\eta(v)$ at three-loop order are

$$\beta(v) = -\varepsilon v + \frac{(N+8)}{6}v^2 - \frac{(3N+14)}{12}v^3 + O(v^4),$$

$$\eta(v) = \frac{(N+2)}{72}v^2\left[1 - \frac{(N+8)}{24}v\right] + O(v^4),$$

$$\eta_2(v) = -\frac{(N+2)}{6}v\left[1 - \frac{5}{12}v\right] + O(v^3).$$

The fixed point value solution of $\beta(v^*) = 0$ is then

$$v^*(\varepsilon) = \frac{6\varepsilon}{(N+8)}\left[1 + \frac{3(3N+14)}{(N+8)^2}\varepsilon\right] + O(\varepsilon^3).$$

The values of the critical exponents

$$\eta = \eta(v^*), \quad \gamma = \frac{2-\eta}{2+\eta_2(v^*)}, \quad \omega = \beta'(v^*),$$

follow

$$\eta = \frac{\varepsilon^2(N+2)}{2(N+8)^2}\left[1 + \frac{(-N^2+56N+272)}{4(N+8)^2}\varepsilon\right] + O(\varepsilon^4),$$
$$\gamma = 1 + \frac{(N+2)}{2(N+8)}\varepsilon + \frac{(N+2)}{4(N+8)^3}\left(N^2+22N+52\right)\varepsilon^2 + O(\varepsilon^3),$$
$$\omega = \varepsilon - \frac{3(3N+14)}{(N+8)^2}\varepsilon^2 + O(\varepsilon^3).$$

Though this may not be obvious on these few terms, the ε-expansion is divergent for any $\varepsilon > 0$, as large order estimates based on instanton calculus have shown. For example, adding simply the known successive terms for $\varepsilon = 1$ and $N = 1$ yields

$$\gamma = 1.000\ldots,\ 1.1666\ldots,\ 1.2438\ldots, 1.1948\ldots,\ 1.3384\ldots,\ 0.8918\ldots,$$

while the best field theory estimate is $\gamma = 1.2396 \pm 0.0013$. Extracting precise numbers from the known terms of the series thus requires a summation method.

Summation of the ε-expansion and numerical values of exponents

We display below (Table 1) the results for the critical exponents γ, ν, η, β and the correction exponent ω of the $O(N)$ model obtained from a Borel summation of the ε-expansion (Guida and Zinn-Justin 1998). Due to scaling relations like $\gamma = \nu(2-\eta)$, $\gamma + 2\beta = \nu d$, only two among the first four are independent, but the series have been summed independently to check consistency and precision.

$N = 0$ corresponds to statistical properties of polymers (mathematically the self-avoiding random walk), $N = 1$, to the Ising universality class, which includes liquid-vapour, binary mixtures or anisotropic magnet phase transitions. $N = 2$ describes the superfluid Helium transition, while $N = 3$ corresponds to isotropic ferromagnets.

Table 1

Critical exponents of the $O(N)$ model, $d = 3$, obtained from the ε-expansion.

N	0	1	2	3
γ	1.1571 ± 0.0030	1.2355 ± 0.0050	1.3110 ± 0.0070	1.3820 ± 0.0090
ν	0.5878 ± 0.0011	0.6290 ± 0.0025	0.6680 ± 0.0035	0.7045 ± 0.0055
η	0.0315 ± 0.0035	0.0360 ± 0.0050	0.0380 ± 0.0050	0.0375 ± 0.0045
β	0.3032 ± 0.0014	0.3265 ± 0.0015	0.3465 ± 0.0035	0.3655 ± 0.0035
ω	0.828 ± 0.023	0.814 ± 0.018	0.802 ± 0.018	0.794 ± 0.018

As a comparison, we also display (Table 2) the best available field theory results obtained from Borel summation of $d = 3$ renormalized perturbative series (Le Guillou and Zinn-Justin 1980, Guida and Zinn-Justin 1998) based on the Callan–Symanzik (CS) formalism, following an initial suggestion of Parisi. In the CS formalism, the renormalized vertex functions are defined by the conditions

$$\tilde{\Gamma}^{(2)}_{\mathrm{r}}(p) = m^2 + p^2 + O(p^4),$$
$$\tilde{\Gamma}^{(4)}_{\mathrm{r}}(0,0,0,0) = m^{4-d} g_{\mathrm{r}} \,.$$

Table 2

Critical exponents of the $O(N)$ model, $d = 3$, obtained from the $(\phi^2)^2_3$ field theory.

N	0	1	2	3
g_{r}^*	26.63 ± 0.11	23.64 ± 0.07	21.16 ± 0.05	19.06 ± 0.05
γ	1.1596 ± 0.0020	1.2396 ± 0.0013	1.3169 ± 0.0020	1.3895 ± 0.0050
ν	0.5882 ± 0.0011	0.6304 ± 0.0013	0.6703 ± 0.0015	0.7073 ± 0.0035
η	0.0284 ± 0.0025	0.0335 ± 0.0025	0.0354 ± 0.0025	0.0355 ± 0.0025
β	0.3024 ± 0.0008	0.3258 ± 0.0014	0.3470 ± 0.0016	0.3662 ± 0.0025
ω	0.812 ± 0.016	0.799 ± 0.011	0.789 ± 0.011	0.782 ± 0.0013

Langevin and Fokker–Planck equations

In the framework of critical phenomena, time evolution is generally described by a phenomenological Langevin equation and its RG properties then govern critical dynamics. However, to a given equilibrium distribution can be associated an infinite number of Langevin equations and, thus, there are many more dynamic than static universality classes.

The Langevin equation is a first order in time stochastic differential equation. For simplicity, we consider here only a purely dissipative Langevin equation for a one-component scalar field $\varphi(t,x)$, t being the time and $x \in \mathbb{R}^d$. It has the general form

$$\dot{\varphi}(t,x) = -\tfrac{1}{2}\Omega \frac{\delta \mathcal{H}}{\delta \varphi(t,x)} + \nu(t,x), \tag{8}$$

where the constant Ω^{-1} provides a time scale.

The functional $\mathcal{H}(\varphi)$ is a time-independent, local euclidean Hamiltonian.

An example is

$$\mathcal{H}(\varphi) = \int \mathrm{d}^d x \left[\tfrac{1}{2}\big(\nabla_x \varphi(x)\big)^2 + V\big(\varphi(x)\big)\right].$$

The noise field $\nu(t,x)$ is a stochastic field which we assume to have the Gaussian local distribution (Gaussian white noise)

$$[\mathrm{d}\rho(\nu)] = [\mathrm{d}\nu]\exp\left[-\int \mathrm{d}t\,\mathrm{d}^d x\,\nu^2(t,x)/2\Omega\right]. \tag{9}$$

It can also be characterized by its one- and two-point functions,

$$\langle \nu(t,x)\rangle = 0\,, \quad \langle \nu(t,x)\nu(t',x')\rangle = \Omega\,\delta(t-t')\delta^{(d)}(x-x').$$

The Fokker–Planck equation.

Given the noise (9), and some initial distribution for the field $\varphi(t,x)$, the Langevin equation generates the time-dependent field distribution

$$P\bigl(t,\varphi(x)\bigr) = \langle \delta\left(\varphi(t,x)-\varphi(x)\right)\rangle_{\nu}\,.$$

From the Langevin equation one derives the evolution equation

$$\dot{P}(\varphi,t) = -\Omega \mathbf{H}_{\mathrm{FP}} P(\varphi,t)\,,$$

where the operator $\mathbf{H}_{\mathrm{FP}}$, the Fokker–Planck Hamiltonian, is given by

$$\mathbf{H}_{\mathrm{FP}}\left(\varphi,\frac{\delta}{\delta\varphi}\right) = -\frac{1}{2}\int \mathrm{d}^d x \frac{\delta}{\delta\varphi(x)}\left[\frac{\delta}{\delta\varphi(x)}+\frac{\delta\mathcal{H}}{\delta\varphi(x)}\right].$$

The change $P=\mathrm{e}^{-\mathcal{H}/2}\,\tilde{P}$ transforms $\mathbf{H}_{\mathrm{FP}}$ into

$$\widetilde{\mathbf{H}}_{\mathrm{FP}}\left(\varphi,\frac{\delta}{\delta\varphi}\right) = \frac{1}{2}\mathbf{A}^{\dagger}\mathbf{A} \text{ with } \mathbf{A} = \frac{\delta}{\delta\varphi(x)}+\frac{1}{2}\frac{\delta\mathcal{H}}{\delta\varphi(x)}\,.$$

From the Fokker–Planck equation, one infers that the Langevin equation (8) together with the noise distribution (9) generates a dynamics that converges toward the equilibrium distribution $\mathrm{e}^{-\mathcal{H}(\varphi)}$ (if it is normalizable). Moreover, one also proves that the purely dissipative Langevin equation is associated to a time evolution with detailed balance.

Application to critical dynamics

Within the theory of phase transitions, the important question is whether and how universal properties generalize to the dynamics. The question can be completely answered in the case of the purely dissipative Langevin equation. The solution to this problem requires constructing an RG also for the dynamics. For this purpose, one has first to understand how the Langevin equation for fields renormalizes.

To discuss renormalization, one has to set up a formalism more directly amenable to the ordinary methods of quantum field theory. This can be done by constructing a field integral representation of the time-dependent φ-field correlation functions in terms of an associated local action, which, in this framework, it is natural to call dynamic action. When the static Hamiltonian $\mathcal{H}(\varphi)$ is renormalizable, one finds that the renormalizations of the static theory together with a time scale renormalization, render the Langevin equation finite.

Time-dependent correlation functions and dynamic action

The generating functional $\mathcal{Z}(J)$ of dynamic correlation functions of the field $\varphi(t,x)$ solution of equation (8), is given by the noise expectation value

$$\begin{aligned}\mathcal{Z}(J) &= \left\langle \exp\left[\int \mathrm{d}^d x\,\mathrm{d}t\, J(t,x)\varphi(t,x)\right]\right\rangle_{\nu},\\ &= \int [\mathrm{d}\nu]\exp\left[-\int \mathrm{d}^d x\,\mathrm{d}t\left(\frac{1}{2\Omega}\nu^2(t,x) - J(t,x)\varphi(t,x)\right)\right].\end{aligned}$$

To impose the Langevin equation, we insert into the field integral the identity

$$\int [\mathrm{d}\varphi]\det \mathbf{M}\prod_{t,x}\delta\left[\dot\varphi(t,x) + (\Omega/2)\delta\mathcal{H}/\delta\varphi - \nu(t,x)\right] = 1\,,$$

where $\mathbf{M}$ is the differential operator,

$$\mathbf{M} = \frac{\delta\,\text{Langevin equation}}{\delta\varphi(t',x')} = \frac{\partial}{\partial t}\delta(t-t')\delta(x-x') + \frac{\Omega}{2}\frac{\delta^2\mathcal{H}}{\delta\varphi(t',x')\delta\varphi(t,x)},$$

The δ-function can immediately be used to integrate over the noise ν:

$$\mathcal{Z}(J) = \int [\mathrm{d}\varphi] \det \mathbf{M} \exp\left[-\int \mathrm{d}^d x\, \mathrm{d}t \left(\frac{1}{2\Omega}\left(\dot{\varphi} + \tfrac{1}{2}\Omega \frac{\delta \mathcal{H}(\varphi)}{\delta \varphi}\right)^2 - J\varphi\right)\right].$$

For a system with a discrete set of degrees of freedom (a $d = 0$ dimensional or a lattice regularized field theory), the determinant can be calculated, using the identity

$$\det \mathbf{M} \propto \exp \operatorname{tr} \ln \left[1 + \left(\frac{\partial}{\partial t}\right)^{-1} \frac{\Omega}{2} \frac{\delta \mathcal{H}}{\delta \varphi \delta \varphi}\right].$$

As a consequence of the causality of the Langevin equation, the inverse of the operator $(\partial/\partial t)\delta(t - t')$ is the kernel $\theta(t - t')$ ($\theta(t)$ is the Heaviside step function). In an expansion in powers of Ω, all terms thus vanish when one takes the trace, but the first one that yields

$$\det M \propto \exp\left\{\theta(0)\frac{\Omega}{2}\int \mathrm{d}t\, \mathrm{d}^d x\, \left.\frac{\delta^2 \mathcal{H}}{\delta\varphi(t,x)\delta\varphi(t,x')}\right|_{x'=x}\right\}.$$

For the undefined quantity $\theta(0)$ we choose $\theta(0) = 1/2$, a choice symmetric in time. The final expression then formally reads

$$
\begin{aligned}
\mathcal{Z}(J) &= \int [\mathrm{d}\varphi] \exp\left[-\mathcal{S}(\varphi) + \int \mathrm{d}^d x \, \mathrm{d}t \, J(t,x)\varphi(t,x)\right], \\
\mathcal{S}(\varphi) &= \frac{1}{2\Omega} \int \mathrm{d}^d x \, \mathrm{d}t \left[\left(\dot{\varphi}(t,x)\right)^2 + \tfrac{1}{4}\Omega^2 \left(\frac{\delta \mathcal{H}(\varphi)}{\delta \varphi(t,x)}\right)^2\right] \\
&\quad - \frac{\Omega}{4} \int \mathrm{d}t \, \mathrm{d}^d x \, \frac{\delta^2 \mathcal{H}(\varphi)}{\delta \varphi(x,t) \delta \varphi(x',t)}\bigg|_{x'=x},
\end{aligned} \tag{10}
$$

where we have expanded the square and integrated the cross term:

$$
\int_{t'}^{t''} \mathrm{d}t \int \mathrm{d}^d x \, \dot{\varphi}(t,x) \frac{\delta \mathcal{H}(\varphi)}{\delta \varphi(t,x)} = \mathcal{H}(\varphi(t'')) - \mathcal{H}(\varphi(t')),
$$

an equation valid inside the field integral only for $\theta(0) = 1/2$.

Finally, the dynamic action can also be directly obtained by expressing the solution $P(\varphi, t)$ of the Fokker–Planck equation as a field integral.

The problem of the determinant

In dimension $d > 0$, the dynamic action is undefined when $\mathcal{H}(\varphi)$ is a local functional because the contribution of the determinant is formally proportional to $\delta^{(d)}(0)$:

$$\ln \det \mathbf{M} \propto \int \mathrm{d}t \, \mathrm{d}^d x \, \frac{\delta^2 \mathcal{H}}{\delta\varphi(t,x)\delta\varphi(x',t)}\bigg|_{x'=x} \propto \delta^{(d)}(0).$$

The determinant has thus to be regularized. We have two choices:

(i) With dimensional regularization, terms like $\delta^{(d)}(0)$ vanish and, therefore, the determinant can be completely omitted. With this convention the expression (10) can be used in practical perturbative calculations.

(ii) However, it is useful to keep this divergent term in some regularized form in order to preserve the geometric structure of the dynamic action. This can be achieved with lattice regularization. These geometric properties will determine the form of the renormalization.

The purely dissipative Langevin equation and supersymmetry

We have explained how to associate to the Langevin or Fokker–Planck equations a dynamic action. Quite generally, the dynamic action has a BRS symmetry. This symmetry and its consequences in the form of WT identities can be used to prove that under some general conditions the structure of the Langevin equation is stable under renormalization.

In the particular example of purely dissipative equations with Gaussian noise, the dynamic action has an additional Grassmann symmetry which, combined with the first one, provides the simplest example of supersymmetry: quantum mechanics supersymmetry. To exhibit supersymmetry, an alternative formalism can be introduced, based on Grassmann coordinates and superfields.

Grassmann coordinates and algebraic properties

To discuss supersymmetry, it is convenient to add to time and space two Grassmann coordinates, θ and $\bar{\theta}$, generators of a Grassmann algebra $\mathfrak{A}$:

$$\theta^2 = \bar{\theta}^2 = 0\,, \quad \theta\bar{\theta} = -\bar{\theta}\theta\,.$$

*Grassmann derivatives.*We also define two linear operations acting on $\mathfrak{A}$:

$$\frac{\partial}{\partial\theta}\,, \quad \frac{\partial}{\partial\bar{\theta}}\,,$$

such that (with $\theta \mapsto \theta_1, \bar{\theta} \mapsto \theta_2$) with the θ_i they form a representation of fermion creation and annihilation operators:

$$\begin{aligned}\frac{\partial}{\partial\theta_i}\frac{\partial}{\partial\theta_j} + \frac{\partial}{\partial\theta_j}\frac{\partial}{\partial\theta_i} &= 0\\ \frac{\partial}{\partial\theta_i}\theta_j + \theta_j\frac{\partial}{\partial\theta_i} &= \delta_{ij}\\ \theta_i\theta_j + \theta_i\theta_j &= 0\,.\end{aligned}$$

We then define definite integrals on $\mathfrak{A}$ by

$$\int \mathrm{d}\theta \equiv \frac{\partial}{\partial\theta}\,, \quad \int \mathrm{d}\bar{\theta} \equiv \frac{\partial}{\partial\bar{\theta}}\,.$$

Then, for example,

$$\int \mathrm{d}\theta \int \mathrm{d}\bar{\theta}\ \mathrm{e}^{\mu\bar{\theta}\theta} = \int \mathrm{d}\theta \int \mathrm{d}\bar{\theta}\,(1+\mu\bar{\theta}\theta) = \mu\,.$$

Grassmann parity. One defines an algebra automorphism on $\mathfrak{A}$ by

$$P(\theta) = -\theta\,, \quad P(\bar{\theta}) = -\bar{\theta}\,.$$

Finally, when the number of generators is even, one can define a 'complex' conjugation in $\mathfrak{A}$ (with the properties of a hermitian conjugation) and a scalar product.

Superfields and covariant derivatives

We introduce a superfield notation

$$\phi(t,x;\bar\theta,\theta) = \varphi(t,x) + \theta\bar c(t,x) + c(t,x)\bar\theta + \theta\bar\theta\bar\varphi(t,x)\,,$$

where $\varphi(t,x)$ and $\bar\varphi(t,x)$ are (scalar) boson fields, $\bar c(t,x)$ and $c(t,x)$ are (spinless) fermion fields.

We define also two Grassmann-type derivatives,

$$\bar{\mathrm{D}} = \frac{\partial}{\partial\bar\theta}, \quad \mathrm{D} = \frac{\partial}{\partial\theta} - \bar\theta\frac{\partial}{\partial t}. \tag{11}$$

$\bar{\mathrm{D}}$ and D satisfy the anticommutation relations

$$\mathrm{D}^2 = \bar{\mathrm{D}}^2 = 0\,, \qquad \mathrm{D}\bar{\mathrm{D}} + \bar{\mathrm{D}}\mathrm{D} = -\frac{\partial}{\partial t}\,. \tag{12}$$

Supersymmetry

The dynamic action corresponding to the Langevin equation

$$\dot{\varphi}(t,x) = -\tfrac{1}{2}\Omega \frac{\delta \mathcal{H}}{\delta \varphi(t,x)} + \nu(t,x),$$

with the noise Gaussian distribution

$$[\mathrm{d}\rho(\nu)] = [\mathrm{d}\nu] \exp\left[-\int \mathrm{d}t\, \mathrm{d}^d x\, \nu^2(t,x)/2\Omega\right],$$

can then be rewritten in supersymmetric form as

$$\mathcal{S}(\phi) = \int \mathrm{d}\bar{\theta}\, \mathrm{d}\theta\, \mathrm{d}t \left[\frac{2}{\Omega} \int \mathrm{d}^d x\, \bar{\mathrm{D}}\phi \mathrm{D}\phi + \mathcal{H}(\phi)\right]. \tag{13}$$

For convenience, we have rescaled the Langevin equation by a factor $2/\Omega$.

We then introduce the two other Grassmann-type differential operators

$$\mathrm{Q} = \frac{\partial}{\partial\theta}, \qquad \bar{\mathrm{Q}} = \frac{\partial}{\partial\bar{\theta}} + \theta\frac{\partial}{\partial t}. \tag{14}$$

Both anticommute with D and $\bar{\mathrm{D}}$ and satisfy

$$\mathrm{Q}^2 = \bar{\mathrm{Q}}^2 = 0\,, \quad \mathrm{Q}\bar{\mathrm{Q}} + \bar{\mathrm{Q}}\mathrm{Q} = \frac{\partial}{\partial t}\,. \tag{15}$$

The two pairs $\mathrm{D}, \bar{\mathrm{D}}$ and $\mathrm{Q}, \bar{\mathrm{Q}}$ provide the simplest examples of generators of supersymmetry. Moreover, Q and $\bar{\mathrm{Q}}$ are generators of symmetries of the dynamic action, as one verifies by performing variations of ϕ of the form

$$\delta\phi(t,\theta,\bar{\theta}) = \varepsilon Q\phi(t,\theta,\bar{\theta}), \quad \delta\phi(t,\theta,\bar{\theta}) = \bar{\varepsilon}\bar{\mathrm{Q}}\phi(t,\theta,\bar{\theta}), \tag{16}$$

where ε and $\bar{\varepsilon}$ are anticommuting constants. The variation of the action density is then a total time derivative. The action is thus supersymmetric.

This confirms that the operators D and $\bar{\mathrm{D}}$ are covariant derivatives from the point of view of supersymmetry.

This supersymmetry is directly related to the property that the corresponding Fokker–Planck Hamiltonian is equivalent to a positive Hamiltonian of the form $\mathbf{A}^{\dagger}\mathbf{A}$.

Remarks.

(i) The anticommutator of $\bar{\mathrm{Q}}$ and Q generates time translations. Supersymmetry implies translation invariance.

(ii) It is possible to emphasize the symmetric role played by $\bar{\theta}$ and θ by performing the substitution $t \mapsto t + \frac{1}{2}\theta\bar{\theta}$.

(iii) Considering the fermions as real dynamic variables, one can associate to the supersymmetric action a Hamiltonian in boson–fermion space. It corresponds then both to the Langevin equation and its time-reversed form.

(iv) Supersymmetry provides a proof of detailed balance, alternative to the proof based on the Fokker–Planck equation.

Ward–Takahashi (WT) identities

One symmetry simply implies that correlation functions are invariant under a translation of the coordinate θ. The second transformation has a slightly more complicated form. It implies that the generating functional $\mathcal{W}(J)$ of connected correlation functions satisfies,

$$\int \mathrm{d}x\, \mathrm{d}t\, \mathrm{d}\bar{\theta}\, \mathrm{d}\theta\, \bar{\mathrm{Q}} J(t,x,\theta,\bar{\theta}) \frac{\delta \mathcal{W}}{\delta J(t,x,\theta,\bar{\theta})} = 0\,.$$

Connected correlation functions $W^{(n)}(t_i, x_i, \theta_i, \bar{\theta}_i)$ and vertex functions $\Gamma^{(n)}(t_i, x_i, \theta_i, \bar{\theta}_i)$ thus satisfy the WT identities:

$$\bar{\mathrm{Q}}\, W^{(n)}(t_i, x_i, \theta_i, \bar{\theta}_i) = 0\,, \qquad \bar{\mathrm{Q}}\, \Gamma^{(n)}(t_i, x_i, \theta_i, \bar{\theta}_i) = 0$$

with

$$\bar{\mathrm{Q}} \equiv \sum_{k=1}^{n} \left(\frac{\partial}{\partial \bar{\theta}_k} + \theta_k \frac{\partial}{\partial t_k} \right).$$

Renormalization of the dissipative Langevin equation

To be more specific, we now assume that $\mathcal{H}(\varphi)$ has the form

$$\mathcal{H}(\varphi) = \frac{1}{2}\int \mathrm{d}^d x (\nabla_x \varphi)^2 + \mathcal{V}(\varphi), \quad \mathcal{V}(\varphi) = \frac{1}{2}m^2\varphi^2 + O(\varphi^3).$$

Then the propagator in the dynamic theory, in the Fourier representation, reads ($\delta(\theta) = \theta$)

$$\tilde{\Delta}(\omega, \mathbf{k}, \boldsymbol{\theta}', \boldsymbol{\theta}) = \frac{\Omega\left[1 + \frac{1}{2}i\omega\left(\theta' - \theta\right)\left(\bar{\theta} + \bar{\theta}'\right) + \frac{1}{4}\Omega\left(k^2 + m^2\right)\delta^2(\bar{\boldsymbol{\theta}}' - \bar{\boldsymbol{\theta}})\right]}{\omega^2 + \frac{\Omega^2}{4}\left(k^2 + m^2\right)^2}.$$

A power counting follows

$$[k] = 1\,,\;\; [\omega] = 2\,,\;\; [\theta] = [\bar{\theta}] = -1\,.$$

Similarly, then (since integration and differentiation over anticommuting variables are identical operations, the dimension of $\mathrm{d}\theta$ is $-[\theta]$)

$$[x] = -1\,,\;\; [t] = -2\,,\;\; [\mathrm{d}\theta] = [\mathrm{d}\bar{\theta}] = 1 \;\Rightarrow [\mathrm{d}t] + [\mathrm{d}\theta] + [\mathrm{d}\bar{\theta}] = 0\,.$$

Therefore, the term proportional to $\mathcal{H}(\phi)$ in the action has the same canonical dimension as in the static case: the static and the dynamic theories have the same power counting and both theories are renormalizable in the same space dimension d.

The supersymmetry transformation is linearly represented on the fields and, therefore, the renormalized action remains supersymmetric. We then write in superfield notation the most general form of the renormalized action $\mathcal{S}_{\mathrm{r}}$ consistent with power counting and supersymmetry:

$$\mathcal{S}_{\mathrm{r}}(\phi) = \int \mathrm{d}\bar{\theta}\,\mathrm{d}\theta\,\mathrm{d}t \left[\frac{2Z_\omega}{\Omega} \int \mathrm{d}^d x\, \bar{\mathrm{D}}\phi \mathrm{D}\phi + \mathcal{H}_{\mathrm{r}}(\phi)\right]. \qquad (17)$$

Only one new renormalization constant, Z_ω, is generated. After renormalization, the drift force in the Langevin equation is thus proportional to the functional derivative of the renormalized Hamiltonian.

The dissipative Langevin equation: RG equations in $d = 4 - \varepsilon$ dimension

We discuss only purely dissipative dynamics (which thus satisfies detailed balance) and, to simplify, without conservation laws. In the classification of the review article of Halperin and Hohenberg, we consider model A. We have described previously most of the technology required to derive RG equations for the dynamics. Note that we will write the RG equations for the renormalized theory, using subscript 0 for the initial parameters.

The N-vector model near four dimensions. We consider the dissipative dynamics for the N-component field $\boldsymbol{\varphi}$,

$$\dot{\boldsymbol{\varphi}}(t,x) = -\frac{\Omega_0}{2}\frac{\delta\mathcal{H}(\boldsymbol{\varphi})}{\delta\boldsymbol{\varphi}(t,x)} + \boldsymbol{\nu}(t,x) \tag{18}$$

with

$$\mathcal{H}(\boldsymbol{\varphi}) = \int \mathrm{d}^d x \left[\frac{1}{2}\left(\partial_\mu\boldsymbol{\varphi}\right)^2 + \frac{1}{2}r_0\boldsymbol{\varphi}^2 + g_0\frac{\Lambda^\varepsilon}{4!}\left(\boldsymbol{\varphi}^2\right)^2\right].$$

The noise field has the Gaussian distribution

$$[\mathrm{d}\rho(\boldsymbol{\nu})] = [\mathrm{d}\nu] \exp\left[-\int \mathrm{d}t\, \mathrm{d}^d x\, \boldsymbol{\nu}^2(t,x)/2\Omega_0\right],$$

in such way that $\mathrm{e}^{-\mathcal{H}}$ is the equilibrium distribution.

In terms of the superfield

$$\boldsymbol{\phi} = \boldsymbol{\varphi} + \theta\bar{\mathbf{c}} + \mathbf{c}\bar{\theta} + \theta\bar{\theta}\bar{\boldsymbol{\varphi}}\,,$$

the corresponding dynamic action $\mathcal{S}(\boldsymbol{\phi})$ takes the supersymmetric form

$$\mathcal{S}(\boldsymbol{\phi}) = \int \mathrm{d}t\, \mathrm{d}\bar{\theta}\, \mathrm{d}\theta \left[\int \mathrm{d}^d x \frac{2}{\Omega_0} \bar{\mathrm{D}}\phi \mathrm{D}\phi + \mathcal{H}(\boldsymbol{\phi})\right]$$

with

$$\bar{\mathrm{D}} = \frac{\partial}{\partial\bar{\theta}}, \quad \mathrm{D} = \frac{\partial}{\partial\theta} - \bar{\theta}\frac{\partial}{\partial t}.$$

RG equations at and above T_c

We have shown that static and supersymmetric dynamic theories have the same upper-critical dimension. Therefore, fluctuations are only relevant for dimensions $d \le 4$. We have also shown that the renormalized dynamic action $\mathcal{S}_{\mathrm{r}}(\boldsymbol{\phi})$ then takes the form

$$\mathcal{S}_{\mathrm{r}}(\boldsymbol{\phi}) = \int \mathrm{d}\bar{\theta}\,\mathrm{d}\theta\,\mathrm{d}t \left[\int \mathrm{d}^d x \frac{2}{\Omega} Z_\omega \bar{\mathrm{D}}\phi \mathrm{D}\phi + \mathcal{H}_{\mathrm{r}}(\boldsymbol{\phi})\right],$$

in which $\boldsymbol{\phi}$ is now the renormalized field and $\mathcal{H}_{\mathrm{r}}(\boldsymbol{\phi})$ is the static renormalized Hamiltonian.

To renormalize the dynamic action, we need, in addition to the static renormalization constants, a renormalization of the parameter Ω:

$$\Omega_0 = \Omega Z / Z_\omega \,,$$

where Z is the field renormalization constant.

The RG differential operator then takes the form (μ is the renormalization scale)

$$\mathrm{D}_{\mathrm{RG}} = \mu\frac{\partial}{\partial\mu} + \beta(g)\frac{\partial}{\partial g} + \eta_\omega(g)\Omega\frac{\partial}{\partial\Omega} - \frac{n}{2}\eta(g),$$

where the new independent RG function $\eta_\omega(g)$ is given by

$$\eta_\omega(g) = \mu\frac{\mathrm{d}}{\mathrm{d}\mu}\bigg|_{g_0,\Omega_0} \ln\Omega\,. \tag{19}$$

The RG equations for the critical theory, in Fourier space and for the vertex functions, then read

$$\left[\mu\frac{\partial}{\partial\mu} + \beta(g)\frac{\partial}{\partial g} + \eta_\omega(g)\Omega\frac{\partial}{\partial\Omega} - \frac{n}{2}\eta(g)\right]\tilde{\Gamma}^{(n)}(p_i,\omega_i,\boldsymbol{\theta},\mu,\Omega,g) = 0\,,$$

where we have used the vector notation $\boldsymbol{\theta} \equiv (\bar{\theta},\theta)$.

At an IR fixed point g^*, they reduce to

$$\left(\mu\frac{\partial}{\partial\mu} + \eta_\omega(g^*)\Omega\frac{\partial}{\partial\Omega} - \frac{n}{2}\eta(g^*)\right)\tilde{\Gamma}^{(n)}(p_i, \omega_i, \boldsymbol{\theta}, \mu, \Omega) = 0\,.$$

We then set $z = 2 + \eta_\omega(g^*)$.

From dimensional analysis, we obtain

$$\tilde{\Gamma}^{(n)}\left(\lambda p_i, \rho\omega_i, \frac{\boldsymbol{\theta}}{\sqrt{\rho}}, \lambda\mu, \frac{\rho\Omega}{\lambda^2}\right) = \lambda^{d-n(d-2)/2}\rho^{1-n}\tilde{\Gamma}^{(n)}(p_i, \omega_i, \boldsymbol{\theta}, \mu, \Omega),$$

In the dimensional equation, we choose $\rho = \Omega\lambda^z\mu^{-\eta_\omega}$. Then, combining the solution of the RG equation with dimensional analysis, we find the dynamic scaling relations

$$\tilde{\Gamma}^{(n)}(\lambda p_i, \omega_i, \boldsymbol{\theta}, \mu = \Omega = 1) = \lambda^{d-nd_\varphi-z(n-1)}F^{(n)}(p_i, \lambda^{-z}\omega_i, \boldsymbol{\theta}\lambda^{z/2}),$$

where $d_\varphi = \frac{1}{2}(d - 2 + \eta)$ is the field dimension.

A few algebraic manipulations yield the corresponding relation for connected correlation functions:

$$\tilde{W}^{(n)}(\lambda p_i, \omega_i, \boldsymbol{\theta}, \mu = \Omega = 1) = \lambda^{(d+z)(1-n)+nd_\varphi} G^{(n)}(p_i, \omega_i \lambda^{-z}, \boldsymbol{\theta}\lambda^{z/2}).$$

The $\boldsymbol{\varphi}$-field two-point correlation function is obtained for $n = 2$ and $\boldsymbol{\theta} = 0$:

$$\tilde{W}^{(2)}(p, \omega, \boldsymbol{\theta} = 0) \sim p^{-2+\eta-z} G^{(2)}(\omega/p^z).$$

The equal-time correlation function is obtained by integrating over ω. One verifies consistency with static scaling.

The dynamic critical two-point function thus depends on a frequency scale that vanishes at small momentum like p^z or a time scale that diverges like p^{-z}.

The RG function η_ω at one-loop order for this model is

$$\eta_\omega(g) = N_d^2 \frac{N+2}{72} \left[6\ln(4/3) - 1\right] g^2 + O\left(\tilde{g}^3\right), \quad N_d = \frac{2}{\Gamma(d/2)(4\pi)^{d/2}}. \quad (20)$$

The dynamic critical exponent z follows:

$$z = 2 + \frac{N+2}{2(N+8)^2}\,[6\ln(4/3)-1]\,\varepsilon^2 + 0\left(\varepsilon^3\right).$$

Correlation functions above T_c in the critical domain. We only write the RG equations at the IR fixed point:

$$\left(\mu\frac{\partial}{\partial\mu} + \eta_\omega\Omega\frac{\partial}{\partial\Omega} - \eta_2\sigma\frac{\partial}{\partial\sigma} - \frac{n}{2}\eta\right)\tilde{\Gamma}^{(n)}(p_i,\omega_i,\boldsymbol{\theta},\sigma,\mu,\Omega) = 0\,, \qquad (21)$$

in which σ is a measure of the deviation from the critical temperature:

$$\sigma \propto T - T_c\,.$$

Dimensional analysis yields

$$\tilde{\Gamma}^{(n)}(p_i,\omega_i,\boldsymbol{\theta},\sigma,\mu,\Omega) = \lambda^{d-n(d-2)/2}\rho^{1-n}\tilde{\Gamma}^{(n)}\left(\frac{p_i}{\lambda},\frac{\omega_i}{\rho},\boldsymbol{\theta}\sqrt{\rho},\frac{\sigma}{\lambda^2},\frac{\mu}{\lambda},\frac{\Omega\lambda^2}{\rho}\right).$$

Finally, combining this equation with the RG equation (21) and choosing

$$\lambda = \sigma^{\nu} \mu^{\nu\eta 2} \sim \xi^{-1}, \qquad \rho = \Omega \mu^{-\eta\omega} \lambda^{z} \sim \xi^{-z},$$

in which ξ is the correlation length, one obtains

$$\begin{aligned} \tilde{\Gamma}^{(n)}(p_i, \omega_i, \boldsymbol{\theta}, \sigma, \mu = 1, \Omega = 1) \sim \xi^{-d+n(d-2+\eta)/2+z(n-1)} \\ \times F^{(n)}(p_i \xi, \omega_i \xi^{z}, \boldsymbol{\theta} \xi^{-z/2}). \end{aligned}$$

Only the combination $\omega_i \xi^z$ appears in the r.h.s. and this implies, after Fourier transformation, that all times are measured in units of a correlation time τ that diverges at the critical temperature as

$$\tau \propto \xi^{z}, \tag{22}$$

an effect called critical slowing down.

Stochastic equations and BRS symmetry

There is a whole set of topics, stochastic field equations, gauge theories, that have a common feature: they involve a constraint equation to which, by a set of formal identities, is associated a quantum action. Indeed, for many problems related to perturbation theory, divergences and renormalization, it is more convenient to work with an action and a field integral rather than with the equation directly, because then standard tools of quantum field theory become available. The quantum action then is invariant under transformations that depend on anticommuting parameters. This symmetry has no geometric origin but is merely an automatic consequence of the construction. It has been first discovered in the context of quantized gauge theories by Becchi, Rouet and Stora, and is called now BRS symmetry.
The generator of the BRS symmetry has a vanishing square and generalizes exterior differentiation. It can be conveniently expressed in compact form by introducing Grassmann coordinates.

We discuss this mathematical structure from a rather formal point of view, using a notation adapted to a finite number of degrees of freedom but the generalization to an infinite number of degrees of freedom is simple. We show that the BRS symmetry is remarkably stable against a number of algebraic transformations.

An especially important class of stochastic equations are Langevin equations. They have been proposed to describe the dynamics of critical phenomena. In all cases divergences appear in perturbative calculations and it is necessary to understand how these equations renormalize. For this purpose one constructs an associated action, often called dynamic action. The BRS symmetry of the dynamic action can be used to prove that under some general conditions the structure of the Langevin equation is stable under renormalization. In the case of purely dissipative Langevin equation, BRS symmetry becomes part of a larger symmetry called supersymmetry.

BRS symmetry and constraint equations

Let φ^α be a set of dynamical variables satisfying a system of equations,

$$E_\alpha(\varphi) = 0\,, \tag{23}$$

where the functions $E_\alpha(\varphi)$ are smooth, and $E_\alpha = E_\alpha(\varphi)$ is a one-to-one mapping in some neighbourhood of $E_\alpha = 0$ which can be inverted in $\varphi^\alpha = \varphi^\alpha(E)$. This implies in particular that equation (23) has a unique solution $\varphi_{\mathrm{s}}^\alpha \equiv \varphi^\alpha(0)$. In the neighbourhood of φ_{s} the determinant $\det \mathbf{E}$ of the matrix $\mathbf{E}$ with elements $E_{\alpha\beta}$,

$$E_{\alpha\beta} \equiv \partial_\beta E_\alpha\,,$$

does not vanish and thus we choose $E_\alpha(\varphi)$ such that it is positive.

Note that it will be convenient throughout the section to use the notation $\partial/\partial\varphi^\alpha \mapsto \partial_\alpha$.

For any function $F(\varphi)$, we now derive a formal expression for $F(\varphi_{\rm s})$ that does not involve solving equation (23) explicitly. We start from the trivial identity

$$F(\varphi_{\rm s}) = \int \left\{ \prod_\alpha \mathrm{d}E^\alpha \, \delta(E_\alpha) \right\} F\big(\varphi(E)\big),$$

where $\delta(E)$ is Dirac's δ-function. We then change variables $E \mapsto \varphi$. This generates the Jacobian $\mathcal{J}(\varphi) = \det \mathbf{E} > 0$. Thus,

$$F(\varphi_{\rm s}) = \int \left\{ \prod_\alpha \mathrm{d}\varphi^\alpha \, \delta\left[E_\alpha(\varphi)\right] \right\} \mathcal{J}(\varphi)\, F(\varphi). \tag{24}$$

We then replace the δ-function by its Fourier representation:

$$\prod_\alpha \delta\left[E_\alpha(\varphi)\right] = \int_{-i\infty}^{+i\infty} \prod_\alpha \frac{\mathrm{d}\bar{\varphi}^\alpha}{2i\pi} \exp\left[\sum_\alpha -\bar{\varphi}^\alpha E_\alpha(\varphi)\right],$$

where the $\bar{\varphi}$ integration runs along the imaginary axis.

Moreover, a determinant can be written as an integral over Grassmann variables $\bar{c}^\alpha$ and c^α:

$$\det \mathbf{E} = \int \prod_\alpha (\mathrm{d}\bar{c}^\alpha \mathrm{d}c^\alpha) \exp\left(\sum_{\alpha,\beta} c^\alpha E_{\alpha\beta} \bar{c}^\beta\right).$$

The expression (24) then becomes

$$F(\varphi_\mathrm{s}) = \mathcal{N} \int \prod_\alpha (\mathrm{d}\varphi^\alpha \mathrm{d}\bar{\varphi}^\alpha \mathrm{d}\bar{c}^\alpha \mathrm{d}c^\alpha)\, F(\varphi) \exp\left[-S(\varphi, \bar{\varphi}, c, \bar{c})\right], \tag{25}$$

in which $S(\varphi, \bar{\varphi}, c, \bar{c})$ is the function (and element of the Grassmann algebra)

$$S(\varphi, \bar{\varphi}, c, \bar{c}) = \sum_\alpha \bar{\varphi}^\alpha E_\alpha(\varphi) - \sum_{\alpha,\beta} c^\alpha E_{\alpha\beta}(\varphi) \bar{c}^\beta \tag{26}$$

and $\mathcal{N}$ the normalization constant determined by setting $F = 1$:

$$\mathcal{N}^{-1} = \int \prod_\alpha (\mathrm{d}\varphi^\alpha \mathrm{d}\bar{\varphi}^\alpha \mathrm{d}\bar{c}^\alpha \mathrm{d}c^\alpha) \exp\left[-S(\varphi, \bar{\varphi}, c, \bar{c})\right].$$

BRS symmetry. Somewhat surprisingly, the function S has a new type of symmetry, a BRS symmetry, first discovered in quantized gauge theories by Becchi, Rouet and Stora. It is a Grassmann symmetry in the sense that the parameter ε of the transformation is an anticommuting constant, an additional generator of the Grassmann algebra. The variations of the various dynamic variables are

$$\delta\varphi^{\alpha} = \varepsilon\bar{c}^{\alpha}\,, \qquad \delta\bar{c}^{\alpha} = 0\,, \tag{27a}$$

$$\delta c^{\alpha} = \varepsilon\bar{\varphi}^{\alpha}\,, \qquad \delta\bar{\varphi}^{\alpha} = 0 \tag{27b}$$

with

$$\varepsilon^2 = 0\,, \qquad \varepsilon\bar{c}^{\alpha} + \bar{c}^{\alpha}\varepsilon = 0\,, \qquad \varepsilon c^{\alpha} + \varepsilon c^{\alpha} = 0\,.$$

The transformation is obviously nilpotent of vanishing square: $\delta^2 = 0$.

The BRS transformation can be represented by a Grassmann differential operator $\mathcal{D}$, when acting on functions of $\{\varphi, \bar{\varphi}, c, \bar{c}\}$:

$$\mathcal{D} = \sum_{\alpha} \left(\bar{c}^{\alpha} \frac{\partial}{\partial \varphi^{\alpha}} + \bar{\varphi}^{\alpha} \frac{\partial}{\partial c^{\alpha}} \right) . \tag{28}$$

The nilpotency of the BRS transformation is then expressed by the identity

$$\mathcal{D}^2 = 0 \, . \tag{29}$$

The differential operator $\mathcal{D}$ is a cohomology operator, generalization of the exterior differentiation of differential forms. In particular, the first term $\sum_{\alpha} \bar{c}^{\alpha} \partial_{\alpha}$ in the BRS operator is identical to the differentiation of forms in a formalism in which the Grassmann variables $\bar{c}^{\alpha}$ are introduced as external variables to exhibit the antisymmetry of the corresponding tensors.

Equation (29) implies that all quantities of the form $\mathcal{D}Q(\varphi, \bar{\varphi}, c, \bar{c})$, quantities we call BRS exact, are BRS invariant. We immediately verify that the function S defined by equation (26) is BRS exact:

$$S = \mathcal{D}\left[\sum_{\alpha} c^{\alpha} E_{\alpha}(\varphi)\right]. \tag{30}$$

It follows that S is BRS invariant,

$$\mathcal{D}S = 0\,. \tag{31}$$

The reciprocal property, the meaning and implications of the BRS symmetry will be discussed in the coming sections.

These properties play an important role, in particular, in the discussion of the renormalization of gauge theories.

Grassmann coordinates, gradient equations

A more compact representation of BRS transformations is obtained by introducing a Grassmann coordinate θ and two functions of θ:

$$\phi^\alpha(\theta) = \varphi^\alpha + \theta\bar{c}^\alpha, \qquad C^\alpha(\theta) = c^\alpha + \theta\bar{\varphi}^\alpha \,. \tag{32}$$

With this notation, the BRS transformations (27) simply become a translation of θ:

$$\begin{aligned} \delta\phi^\alpha(\theta) &= \varepsilon\frac{\partial\phi^\alpha}{\partial\theta} = \phi^\alpha(\theta+\varepsilon) - \phi^\alpha(\theta), \\ \delta C^\alpha(\theta) &= \varepsilon\frac{\partial C^\alpha}{\partial\theta} = C^\alpha(\theta+\varepsilon) - C^\alpha(\theta). \end{aligned} \tag{33}$$

In particular, the BRS operator $\mathcal{D}$ is represented by $\partial/\partial\theta$:

$$\mathcal{D} \mapsto \frac{\partial}{\partial\theta} \,.$$

From the expansion

$$\sum_{\alpha} C^{\alpha}(\theta) E_{\alpha}(\phi(\theta)) = \sum_{\alpha} c^{\alpha} E_{\alpha}(\varphi) + \theta \left[\sum_{\alpha} \bar{\varphi}^{\alpha} E_{\alpha}(\varphi) - \sum_{\alpha,\beta} c^{\alpha} \frac{\partial E_{\alpha}}{\partial \varphi^{\beta}} \bar{c}^{\beta} \right],$$

we recover equation (30) in a different notation:

$$S(\varphi, \bar{\varphi}, c, \bar{c}) = \frac{\partial}{\partial \theta} \left[C^{\alpha}(\theta) E_{\alpha}(\phi(\theta)) \right]. \tag{34}$$

In the case of Grassmann variables integration and differentiation are identical operations. Therefore, the equation can be rewritten as

$$S(\varphi, \bar{\varphi}, c, \bar{c}) = \int \mathrm{d}\theta\, C^{\alpha}(\theta) E_{\alpha}(\phi(\theta)). \tag{35}$$

In this expression the BRS symmetry is manifest: the integrand does not depend on θ explicitly.

Note that since the function S involves only a Grassmann combination of the form $c\bar{c}$, in a representation in terms of the functions (32), as in equation (35), each integration over θ is associated with a factor $C^{\alpha}(\theta)$.

Gradient equations. In general, the two Grassmann variables $\bar{c}^{\alpha}$ and c^{α} play different roles. However, there is one special situation in which a symmetry is established between them—when the matrix $E_{\alpha\beta}$ is symmetric:

$$E_{\alpha\beta} = E_{\beta\alpha} \iff \partial_{\beta} E_{\alpha} = \partial_{\alpha} E_{\beta} \,.$$

Hence, in the absence of topological obstructions, there exists a function $A(\varphi)$ such that

$$E_{\alpha}(\varphi) = \partial_{\alpha} A(\varphi). \tag{36}$$

The symmetry between c and $\bar{c}$ generates an additional independent BRS symmetry of generator

$$\bar{\mathcal{D}} = \sum_{\alpha} \left(c^{\alpha} \frac{\partial}{\partial \varphi^{\alpha}} + \bar{\varphi}^{\alpha} \frac{\partial}{\partial \bar{c}^{\alpha}} \right).$$

Introducing now two Grassmann coordinates $\bar{\theta}$ and θ (and then $\bar{\mathcal{D}} \mapsto \partial/\partial\bar{\theta}$), and a function

$$\phi^{\alpha}(\bar{\theta}, \theta) = \varphi^{\alpha} + \theta \bar{c}^{\alpha} + c^{\alpha} \bar{\theta} + \theta \bar{\theta} \bar{\varphi}^{\alpha} \,. \tag{37}$$

one verifies that the function $S(\phi)$ then takes the remarkable form

$$S(\phi) = \int \mathrm{d}\bar{\theta}\, \mathrm{d}\theta\, A\left[\phi(\bar{\theta}, \theta)\right] = \bar{\mathcal{D}} \mathcal{D} A(\varphi).$$

The two symmetries, which correspond to independent translations of θ and $\bar{\theta}$, are here explicit.

Stochastic equations

We now assume that equation (23) depends on a set of stochastic variables ν_a, the "noise", with normalized probability distribution $\mathrm{d}\rho(\nu)$:

$$E_\alpha(\varphi,\nu)=0\,. \tag{38}$$

The solution φ^α of the equation becomes a stochastic variable. Quantities of interest are now expectation values of functions of φ:

$$\begin{aligned}\langle F(\varphi)\rangle_\nu &= \int \mathrm{d}\rho(\nu)\prod_\alpha \mathrm{d}\varphi^\alpha\,\delta\left[E_\alpha(\varphi,\nu)\right]\det\mathbf{E}\,F(\varphi)\\ &\propto \int \mathrm{d}\rho(\nu)\prod_\alpha(\mathrm{d}\varphi^\alpha\mathrm{d}\bar\varphi^\alpha\mathrm{d}\bar c^\alpha\mathrm{d}c^\alpha)\,F(\varphi)\exp\left[-S(\varphi,\bar\varphi,c,\bar c,\nu)\right]\end{aligned}$$

with S given by equation (26):

$$S=\sum_\alpha\bar\varphi^\alpha E_\alpha(\varphi,\nu)-\sum_{\alpha,\beta}c^\alpha E_{\alpha\beta}(\varphi,\nu)\bar c^\beta\,. \tag{39}$$

Let us introduce the function $\Sigma(\varphi, \bar{\varphi}, c, \bar{c})$ obtained after noise averaging:

$$\langle F(\varphi)\rangle \propto \int \prod_{\alpha} (\mathrm{d}\varphi^{\alpha} \mathrm{d}\bar{\varphi}^{\alpha} \mathrm{d}\bar{c}^{\alpha} \mathrm{d}c^{\alpha})\, F(\varphi) \exp\left[-\Sigma(\varphi, \bar{\varphi}, c, \bar{c})\right]$$

with

$$\exp\left[-\Sigma(\varphi, \bar{\varphi}, c, \bar{c})\right] = \int \mathrm{d}\rho(\nu) \exp\left[-S(\varphi, \bar{\varphi}, c, \bar{c}, \nu)\right]. \tag{40}$$

We have shown that S has a BRS symmetry. Applying the BRS operator (28) on both sides of equation (40), we conclude that $\Sigma(\varphi, \bar{\varphi}, c, \bar{c})$ is still BRS symmetric,

$$\mathcal{D}\Sigma = 0\,, \tag{41}$$

although it no longer has the simple form (39), that is, a function linear in $\bar{\varphi}$ and $c\bar{c}$. Moreover, because S is BRS exact, the function Σ is also BRS exact, as can be shown by simple algebraic manipulations based on the identity

$$f(\mathcal{D}X) = f(0) + \mathcal{D}\left[X g(\mathcal{D}X)\right] \quad \text{with} \quad g(x) = \frac{f(x) - f(0)}{x}\,.$$

A simple example: Stochastic equations linear in the noise

Stochastic equations of the simple algebraic form

$$E_\alpha(\nu,\varphi) \equiv E_\alpha(\varphi) - \nu_\alpha\,, \tag{42}$$

are often met. Introducing the Laplace transform of the measure $\mathrm{d}\rho(\nu)$,

$$\mathrm{e}^{w(\bar\varphi)} = \int \mathrm{d}\rho(\nu)\exp\left[\sum_\alpha \bar\varphi^\alpha \nu_\alpha\right],$$

we obtain for the function Σ defined by equation (40),

$$\Sigma(\varphi,\bar\varphi,c,\bar c) = -w(\bar\varphi) + \sum_\alpha \bar\varphi^\alpha E_\alpha(\varphi) - \sum_{\alpha,\beta} c^\alpha E_{\alpha\beta}\bar c^\beta \tag{43a}$$

$$= \mathcal{D}\tilde\Sigma\,, \quad \tilde\Sigma = \sum_\alpha c^\alpha\left[E_\alpha(\varphi) - \frac{\partial}{\partial\bar\varphi^\alpha}\int_0^1 \mathrm{d}s\, w(s\bar\varphi)\right]. \tag{43b}$$

Remarks.

(i) After integration over the noise, the expression of the function Σ in the notation of Grassmann coordinates is, in general, rather complicated. However, in the case of equation (42) with Gaussian noise the additional term $w(\bar{\varphi}) = \frac{1}{2} \sum_{\alpha,\beta} w_{\alpha\beta} \bar{\varphi}^{\alpha} \bar{\varphi}^{\beta}$ is represented in the notation (37) by

$$\frac{1}{2} \sum_{\alpha,\beta} w_{\alpha\beta} \bar{\varphi}^{\alpha} \bar{\varphi}^{\beta} = \int \mathrm{d}\bar{\theta} \mathrm{d}\theta \, \frac{1}{2} \sum_{\alpha,\beta} w_{\alpha\beta} \frac{\partial \phi^{\alpha}}{\partial \bar{\theta}} \frac{\partial \phi^{\beta}}{\partial \theta} .$$

(ii) In the latter case, it is also possible to integrate explicitly over the $\bar{\varphi}$ variables. The resulting integrand corresponds to

$$\Sigma(\varphi, c, \bar{c}) = \tfrac{1}{2} \sum_{\alpha,\beta} \left[E_{\alpha}(\varphi) (w^{-1})^{\alpha\beta} E_{\beta}(\varphi) - c^{\alpha} E_{\alpha\beta}(\varphi) \bar{c}^{\beta} \right] .$$

The BRS transformation of c is now non-linear:

$$\delta_{\mathrm{BRS}} c^\alpha = \varepsilon \sum_\beta [w^{-1}]^{\alpha\beta} E_\beta(\varphi) \, .$$

We note that in this form the BRS transformation has a vanishing square only when φ is a solution of the equation $E(\varphi) = 0$. We conclude that the property $\mathcal{D}^2 = 0$ of BRS transformations is not true in all formulations and may be satisfied only after the introduction of some auxiliary variables.

BRS cohomology. We have shown that for a general stochastic equation linear in a Gaussian noise, the weight function Σ obtained after noise averaging is BRS exact and quadratic in $\{\bar\varphi, c\bar c\}$. Conversely, one may ask the question: what is the most general form of a function Σ quadratic in $\{\bar\varphi, c\bar c\}$ and BRS symmetric. BRS cohomology techniques allow showing that in the case of simply connected manifolds, any BRS symmetric function is BRS exact, up to a constant:

$$\mathcal{D}\Sigma = 0 \ \Rightarrow\ \Sigma = \mathcal{D}\widetilde{\Sigma} + \mathrm{const.}\, .$$

Functional (or exact) renormalization group

We now briefly describe a general approach to the RG close to ideas initially developed by Wegner and Wilson, and based on a partial integration over the large-momentum modes of fields. This RG takes the form of functional renormalization group (FRG) equations that express the equivalence between a change of a scale parameter related to microscopic physics and a change of the parameters of the Hamiltonian. Some forms of these equations are exact and one then also speaks of the exact renormalization group.

These FRG equations have been used to recover the first terms of the ε-expansion and later by Polchinski to give a new proof of the renormalizability of field theories, avoiding arguments based on Feynman diagrams and combinatorics.

From the practical viewpoint, several variants of these FRG equations have led to new approximation schemes no longer based on the standard perturbative expansion.

Technically, these FRG equations follow from identities that express the invariance of the partition function under a correlated change of the propagator and the other parameters of the Hamiltonian. We discuss these equations, in continuum space, in the framework of quasi-local statistical field theory (*i.e.*, non-locality is restricted to short distance). It is easy to verify that, except in the Gaussian case, these equations are closed only if an infinite number of local interactions are included.

It is then possible to infer various RGE satisfied by correlation functions. Depending on the chosen form, these RGE are either exact or only exact at large distance or small momenta, up to corrections decaying faster than any power of the dilatation parameter.

Here, we discuss only the Hamiltonian flow, the RGE for correlation functions requiring some additional considerations.

General effective or Landau–Ginzburg–Wilson Hamiltonian

Here, for simplicity we consider a statistical field theory involving only one scalar field $\phi(x)$ in d dimensions. It is defined by a quasi-local (a concept we define below), translation invariant functional $\mathcal{H}(\phi)$ called *Hamiltonian* in the context of statistical physics, which is also a generalization of the *Euclidean action*, that is, the classical action in imaginary time, of the quantum theory of fundamental interactions.

We assume that $\mathcal{H}(\phi)$ has the general properties of the thermodynamic potential of Landau's theory:

(i) $\mathcal{H}(\phi)$ is a regular function of all thermodynamic parameters like the temperature (except at zero temperature).

(ii) It is expandable in powers of the field ϕ:

$$\mathcal{H}(\phi) = \sum_{n=0} \frac{1}{n!} \int \mathrm{d}^d x_1 \mathrm{d}^d x_2 \ldots \mathrm{d}^d x_n \, \mathcal{H}^{(n)}(x_1, x_2, \ldots, x_n) \phi(x_1) \ldots \phi(x_n).$$

(iii) It is translation-invariant and thus

$$\mathcal{H}^{(n)}(x_1, x_2, \ldots, x_n) = \mathcal{H}^{(n)}(x_1 + a, x_2 + a, \ldots, x_n + a) \quad \forall a \in \mathbb{R}^d.$$

Then, the Fourier transforms of the coefficients (functions or distributions) $\mathcal{H}^{(n)}$ take the form

$$(2\pi)^d \delta^{(d)}\left(\sum_{i=1}^{n} p_i\right) \tilde{\mathcal{H}}^{(n)}(p_1, \ldots, p_n) = \int \mathrm{d}^d x_1 \ldots \mathrm{d}^d x_n \, \exp\left(i \sum_{j=1}^{n} x_j p_j\right) \times \mathcal{H}^{(n)}(x_1, \ldots, x_n). \tag{44}$$

(iv) The hypothesis of short-range interactions (with exponential decay), or of quasi-locality, is equivalent to the hypothesis that the coefficients $\tilde{\mathcal{H}}^{(n)}$ are analytic in strips of the form $|\,\mathrm{Im}\, p_i| < \kappa$. This implies that $\mathcal{H}(\phi)$ can be expanded in powers of the field and its derivatives.

In terms of the Fourier components of the field,

$$\phi(x) = \int \mathrm{d}^d k \; \mathrm{e}^{ikx} \, \tilde{\phi}(k),$$

the expansion of $\mathcal{H}$ reads

$$\begin{aligned} \mathcal{H}(\phi) = \sum_{n=0} \frac{1}{n!} \int & \mathrm{d}^d k_1 \ldots \mathrm{d}^d k_n \\ & \times (2\pi)^d \delta^{(d)} \left(\sum_{i=1}^{n} k_i \right) \tilde{\mathcal{H}}^{(n)}(k_1, \ldots, k_n) \tilde{\phi}(k_1) \ldots \tilde{\phi}(k_n). \end{aligned} \tag{45}$$

Remark. One may wonder why one immediately considers such a general class of Hamiltonians, whereas one seems unable to determine the critical behaviour of much simpler systems. Of course, in this way the analysis will apply to a larger class of systems. But the main reason is, as we shall see, that RG transformations generate such Hamiltonians, even when the initial Hamiltonian is much simpler.

Partial field integration and effective Hamiltonian

Using identities that involve only Gaussian integrations, one first proves equality between two partition functions corresponding to two different Hamiltonians. This relation can then be interpreted as resulting from a partial integration over some components of the fields. One infers a sufficient condition for correlated modifications of the propagator and interactions, in a statistical field theory, to leave the partition function invariant.

In what follows, we assume that all Gaussian two-point functions, or propagators, $\Delta(x-y)$ are such that in the Fourier representation

$$\Delta(x) = \frac{1}{(2\pi)^d} \int \mathrm{d}^d p \; \mathrm{e}^{ipx} \, \tilde{\Delta}(p),$$

$\tilde{\Delta}(p)$ decreases exponentially for $|p| \to \infty$ so that the Gaussian expectation value of any local polynomials in the field exists.

Partial integration

One first establishes a relation between partition functions corresponding to two different quasi-local Hamiltonians.

The first Hamiltonian depends on a field ϕ and we write it in the form

$$\mathcal{H}_1(\phi) = \frac{1}{2}\int \mathrm{d}^d x\, \mathrm{d}^d y\, \phi(x) K_1(x-y)\phi(y) + \mathcal{V}_1(\phi), \tag{46}$$

where K_1 is the kernel of a positive operator and the functional $\mathcal{V}_1(\phi)$ is expandable in powers of the field ϕ and translation invariant. We introduce the inverse Δ_1 of K_1:

$$\int \mathrm{d}^d z\, K_1(x-z)\Delta_1(z-y) = \delta^{(d)}(x-y).$$

The second Hamiltonian depends on two fields ϕ_1, ϕ_2 in the form

$$\begin{aligned}\mathcal{H}(\phi_1,\phi_2) = \frac{1}{2}\int \mathrm{d}^d x\, \mathrm{d}^d y\, &\left[\phi_1(x) K_2(x-y)\phi_1(y) + \phi_2(x)\mathcal{K}(x-y)\phi_2(y)\right] \\ &+ \mathcal{V}_1(\phi_1+\phi_2).\end{aligned}$$

Similarly, we introduce

$$\int \mathrm{d}^d z\, K_2(x-z)\Delta_2(z-y) = \delta^{(d)}(x-y), \quad \int \mathrm{d}^d z\, \mathcal{K}(x-z)\mathcal{D}(z-y) = \delta^{(d)}(x-y).$$

We assume that the kernels K_1, K_2 and $\mathcal{K}$ (and thus Δ_1, Δ_2 and $\mathcal{D}$) all correspond to positive operators, a condition that ensures that the field integrals exist at least as formal series in powers of the interaction $\mathcal{V}_1$.

Then, if $\Delta_1 = \Delta_2 + \mathcal{D} \;\Rightarrow\; K_1 = K_2(K_2+\mathcal{K})^{-1}\mathcal{K}$, the ratio of the partition functions

$$\mathcal{Z}_1 = \int [\mathrm{d}\phi]\, \mathrm{e}^{-\mathcal{H}_1(\phi)} \text{ and } \mathcal{Z}_2 = \int [\mathrm{d}\phi_1\, \mathrm{d}\phi_2]\, \mathrm{e}^{-\mathcal{H}(\phi_1,\phi_2)} \tag{47}$$

does not depend on $\mathcal{V}_1$ since

$$\mathcal{Z}_2 = \left(\frac{\det(\mathcal{D}\Delta_2)}{\det \Delta_1}\right)^{1/2} \mathcal{Z}_1\,. \tag{48}$$

A more useful form of the identity. In what follows, we use the compact notation

$$\int \mathrm{d}^d x\, \mathrm{d}^d y\, \phi(x) K(x-y) \phi(y) \equiv (\phi K \phi).$$

We define

$$\mathrm{e}^{-\mathcal{V}_2(\phi)} = (\det \mathcal{D})^{-1/2} \int [\mathrm{d}\varphi] \exp\left[-\tfrac{1}{2}(\varphi \mathcal{K} \varphi) - \mathcal{V}_1(\phi+\varphi)\right], \qquad (49)$$

as well as $\mathcal{H}_2(\phi) = \frac{1}{2}(\phi K_2 \phi) + \mathcal{V}_2(\phi)$. Then, the equivalence takes the more directly useful form

$$\int [\mathrm{d}\phi]\, \mathrm{e}^{-\mathcal{H}_2(\phi)} = \left(\frac{\det \Delta_2}{\det \Delta_1}\right)^{1/2} \int [\mathrm{d}\phi]\, \mathrm{e}^{-\mathcal{H}_1(\phi)}. \qquad (50)$$

The left hand side can be interpreted as resulting from a partial integration over the field ϕ since the propagator $\mathcal{D}$ is positive and, thus, in the sense of operators, $\Delta_2 < \Delta_1$.

Differential form

We now assume that the propagator Δ is a smooth function of a real parameter s, $\Delta \equiv \Delta(s)$, with a negative derivative. We define

$$D(s) = \frac{\mathrm{d}\Delta(s)}{\mathrm{d}s} < 0\,,$$

where $D(s)$ is represented by the kernel $D(s; x - y)$.

For $s < s'$, we identify

$$\Delta_1 = \Delta(s), \quad \Delta_2 = \Delta(s') \quad \text{and thus} \quad \mathcal{D}(s, s') = \Delta(s) - \Delta(s') > 0\,. \tag{51}$$

Similarly,

$$K_1 = K(s) = \Delta^{-1}(s), \quad K_2 = K(s'), \quad \mathcal{K}(s, s') = [\mathcal{D}(s, s')]^{-1} > 0\,. \tag{52}$$

Since the kernels $K(s)$, $\mathcal{K}(s, s')$ are positive, all Gaussian integrals are defined.

Finally, we set

$$\mathcal{V}_1(\phi) = \mathcal{V}(\phi, s), \quad \mathcal{V}_2(\phi) = \mathcal{V}(\phi, s'), \quad \mathcal{H}_1(\phi) = \mathcal{H}(\phi, s), \quad \mathcal{H}_2(\phi) = \mathcal{H}(\phi, s').$$

The equivalence then takes the form

$$\int [\mathrm{d}\phi]\, \mathrm{e}^{-\mathcal{H}(\phi,s')} = \left(\frac{\det \Delta(s')}{\det \Delta(s)}\right)^{1/2} \int [\mathrm{d}\phi]\, \mathrm{e}^{-\mathcal{H}(\phi,s)},$$

where $\mathcal{V}(\phi, s')$ is given by

$$\mathrm{e}^{-\mathcal{V}(\phi,s')} = \big(\det \mathcal{D}(s,s')\big)^{-1/2} \int [\mathrm{d}\varphi] \exp\left[-\tfrac{1}{2}\big(\varphi \mathcal{K}(s,s')\varphi\big) - \mathcal{V}(\phi+\varphi, s)\right]. \tag{53}$$

Differential form

Setting $s' = s + \sigma$, $\sigma > 0$, one expands in powers of $\sigma \to 0$. Identifying the terms of order σ, after some algebraic manipulations one obtains the functional equation

$$\frac{\mathrm{d}}{\mathrm{d}s}\mathcal{V}(\phi, s) = -\frac{1}{2}\int \mathrm{d}^d x\, \mathrm{d}^d y\, D(s; x-y)\left[\frac{\delta^2\mathcal{V}}{\delta\phi(x)\delta\phi(y)} - \frac{\delta\mathcal{V}}{\delta\phi(x)}\frac{\delta\mathcal{V}}{\delta\phi(y)}\right]. \quad (54)$$

The equation expresses a sufficient condition for the partition function

$$\mathcal{Z}(s) = (\det\Delta(s))^{-1/2}\int [\mathrm{d}\phi]\, \mathrm{e}^{-\mathcal{H}(\phi,s)} \text{ with}$$
$$\mathcal{H}(\phi, s) = \tfrac{1}{2}(\phi K(s)\phi) + \mathcal{V}(\phi, s), \quad (55)$$

to be independent of the parameter s.

This property relates a modification of the propagator to a modification of the interaction, quite in the spirit of the RG.

Remark.

(i) A sufficient condition for $\mathcal{Z}(s)$ to be independent of s, is that the equation is satisfied as an expectation value with the measure $\mathrm{e}^{-\mathcal{H}(\phi,s)}$. One can thus add to the equation contributions with vanishing expectation value to derive other sufficient conditions (useful for correlation functions).

(ii) Let us set

$$\Sigma(\phi, s) = \mathrm{e}^{-\mathcal{V}(\phi,s)} .$$

Then, the functional equation reduces to

$$\frac{\mathrm{d}}{\mathrm{d}s}\Sigma(\phi, s) = -\frac{1}{2}\int \mathrm{d}^d x\, \mathrm{d}^d y\, D(s; x-y)\frac{\delta^2\Sigma(\phi, s)}{\delta\phi(x)\delta\phi(y)}.$$

This a functional heat equation since the kernel $-D$ is positive. One may wonder why we do not work with this equation, which is linear and thus much simpler. The reason is that $\mathcal{V}(\phi, s)$ is a local functional, in contrast with $\Sigma(\phi, s)$, and locality that plays an essential role.

Hamiltonian evolution

From the evolution equation, for $\mathcal{V}(\phi, s)$ one then infers the evolution of the Hamiltonian

$$\mathcal{H}(\phi, s) = \tfrac{1}{2}(\phi K(s)\phi) + \mathcal{V}(\phi, s).$$

Introducing the operator $L(s)$, with kernel $L(s; x-y)$, defined by

$$L(s) \equiv D(s)\Delta^{-1}(s) = \frac{\mathrm{d}\ln\Delta(s)}{\mathrm{d}s}, \tag{56}$$

one finds

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}s}\mathcal{H}(\phi, s) = -\frac{1}{2}\int \mathrm{d}^d x\, \mathrm{d}^d y\, D(s; x-y)\left[\frac{\delta^2\mathcal{H}}{\delta\phi(x)\delta\phi(y)} - \frac{\delta\mathcal{H}}{\delta\phi(x)}\frac{\delta\mathcal{H}}{\delta\phi(y)}\right] \\ - \int \mathrm{d}^d x\, \mathrm{d}^d y\, \phi(x) L(s; x-y)\frac{\delta\mathcal{H}}{\delta\phi(y)} + \frac{1}{2}\operatorname{tr} L(s).\end{aligned} \tag{57}$$

Formal solution

Since the evolution equation is a first-order differential equation in s, the form of the functional $\mathcal{V}(\phi, s)$ for an initial value s_0 of the parameter s determines the solution for all $s \geq s_0$. From the very proof of equation (54), its solution can be easily inferred:

$$\mathrm{e}^{-\mathcal{V}(\phi,s)} = \left(\det \mathcal{D}(s_0, s)\right)^{-1/2} \int [\mathrm{d}\varphi] \exp\left[-\tfrac{1}{2}\left(\varphi \mathcal{K}(s_0, s)\varphi\right) - \mathcal{V}(\phi+\varphi, s_0)\right]. \tag{58}$$

The equation implies that $-\mathcal{V}(\phi, s)$ is the sum of the connected contributions of the diagrams constructed with the propagator $\mathcal{K}^{-1}(s_0, s) = \Delta(s) - \Delta(s_0)$ and interactions $\mathcal{V}(\phi + \varphi, s_0)$.

Only if initially $\mathcal{V}(\phi, s_0)$ is a quadratic form (a Gaussian model) it remains so. However, if one adds, for example, a quartic term, then an infinite number of terms of higher degrees are automatically generated.

Field renormalization

In order to be able to find RG fixed-point solutions, it is necessary to introduce a field renormalization. To prove the corresponding identities, we set

$$\phi(x) = \sqrt{Z(s)}\phi'(x), \quad \mathcal{H}(\phi, s) = \mathcal{H}'(\phi', s),$$

where $Z(s)$ is an arbitrary differentiable function. We then define

$$\eta(s) = \frac{\mathrm{d} \ln Z(s)}{\mathrm{d}s} \,.$$

One then infers from the Hamiltonian flow equation the modified equation (omitting some constant term)

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}s}\mathcal{H}(\phi, s) &= -\frac{1}{2}\int \mathrm{d}^d x \, \mathrm{d}^d y \, D(x-y) \left[\frac{\delta^2 \mathcal{H}}{\delta\phi(x)\delta\phi(y)} - \frac{\delta \mathcal{H}}{\delta\phi(x)} \frac{\delta \mathcal{H}}{\delta\phi(y)}\right] \\ &\quad - \int \mathrm{d}^d x \, \mathrm{d}^d y \, \phi(x) L(s; x-y) \frac{\delta \mathcal{H}}{\delta\phi(y)} + \frac{1}{2}\eta(s) \int \mathrm{d}^d x \, \phi(x) \frac{\delta \mathcal{H}}{\delta\phi(x)} \,. \end{aligned} \tag{59}$$

High-momentum mode integration and RGE

These equations can be applied to a situation where the partial integration over the field corresponds, in the Fourier representation, to a partial integration over its high-momentum modes, which in position space also corresponds to an integration over short-distance degrees of freedom.

In what follows, we specialize Δ to a critical (massless) propagator. A possible deviation from the critical theory is included in $\mathcal{V}(\phi)$.

Cut-off parameter and propagator. In the preceding formalism, we now identify $s \equiv -\ln \Lambda$, where Λ is a large-momentum cut-off, which also represents the inverse of the microscopic scale. A variation of s then corresponds to a dilatation of the parameter Λ.

We then choose a regularized propagator Δ_Λ of the form

$$\Delta_\Lambda(x) = \int \frac{\mathrm{d}^d k}{(2\pi)^d}\, \mathrm{e}^{-ikx}\, \tilde{\Delta}_\Lambda(k) \quad \text{with} \quad \tilde{\Delta}_\Lambda(k) = \frac{C(k^2/\Lambda^2)}{k^2}\,.$$

The function $C(t)$ is positive, decreasing, analytic for $|\,\mathrm{Im}\, t| \le \tau$ $(\tau > 0)$, goes to 1 for $t \to 0$ and vanishes faster than any power for $t \to \infty$. In the field integral, it suppresses the contributions of the field Fourier components corresponding to momenta much higher than Λ.

The Fourier transform of the derivative $D_\Lambda(x)$,

$$\tilde{D}_\Lambda(k) = -\Lambda \frac{\partial \tilde{\Delta}_\Lambda(k)}{\partial \Lambda} = \frac{2}{\Lambda^2} C'(k^2/\Lambda^2), \tag{60}$$

has an essential property: it has no pole at $k = 0$ and thus is not critical.

The function

$$D_\Lambda(x) = -\Lambda \frac{\partial \Delta_\Lambda(x)}{\partial \Lambda} = \int \frac{\mathrm{d}^d k}{(2\pi)^d}\, \mathrm{e}^{ikx}\, \tilde{D}_\Lambda(k) = \Lambda^{d-2} D_{\Lambda=1}(\Lambda x), \qquad (61)$$

thus decays for $|x| \to \infty$ exponentially since $C(t)$ is analytical. The propagator $\mathcal{D}(\Lambda_0, \Lambda) = D_{\Lambda_0} - D_\Lambda$, $\Lambda_0 > \Lambda$, whose inverse now appears in the field integral solution of the flow equation, shares this property.

RGE

With these assumptions and definitions, the flow equation for $\mathcal{V}(\phi,\Lambda)$ becomes

$$\Lambda \frac{\mathrm{d}}{\mathrm{d}\Lambda}\mathcal{V}(\phi,\Lambda) = \frac{1}{2}\int \mathrm{d}^d x\, \mathrm{d}^d y\, D_\Lambda(x-y)\left[\frac{\delta^2 \mathcal{V}}{\delta\phi(x)\delta\phi(y)} - \frac{\delta\mathcal{V}}{\delta\phi(x)}\frac{\delta\mathcal{V}}{\delta\phi(y)}\right]. \quad (62)$$

This equation being exact, one uses also the terminology exact renormalization group.

Remarks.

Since $D_\Lambda(x)$ decreases exponentially, if $\mathcal{V}(\phi)$ is initially local, it remains local, a property that becomes more apparent when one expands the equation in powers of ϕ.

The equation differs from the abstract RGE by the property that the scale parameter Λ appears explicitly through the function D_Λ. We shall later eliminate this explicit dependence.

Field Fourier components

In terms of the Fourier components $\tilde{\phi}(k)$ of the field,

$$\phi(x) = \int \mathrm{d}^d k \ \mathrm{e}^{ikx}\, \tilde{\phi}(k) \ \Leftrightarrow \ \tilde{\phi}(k) = \int \frac{\mathrm{d}^d x}{(2\pi)^d}\, \mathrm{e}^{-ikx}\, \phi(x),$$

the equation becomes

$$\Lambda \frac{\mathrm{d}}{\mathrm{d}\Lambda} \mathcal{V}(\phi, \Lambda) = \frac{1}{2} \int \frac{\mathrm{d}^d k}{(2\pi)^d} \tilde{D}_\Lambda(k) \left[\frac{\delta^2 \mathcal{V}}{\delta\tilde{\phi}(k)\delta\tilde{\phi}(-k)} - \frac{\delta\mathcal{V}}{\delta\tilde{\phi}(k)} \frac{\delta\mathcal{V}}{\delta\tilde{\phi}(-k)} \right]. \quad (63)$$

In the equation, locality translates into regularity of the Fourier components. If the coefficients of the expansion of $\mathcal{V}(\phi, \Lambda)$ in powers of $\tilde{\phi}$, after factorization of the δ-function, are regular functions for an initial value of $\Lambda = \Lambda_0$, they remain for $\Lambda < \Lambda_0$ because $\tilde{D}_\Lambda(k)$ is a regular function of k.

Finally, the flow equation for the complete Hamiltonian expressed in terms of Fourier components,

$$\mathcal{H}(\phi, \Lambda) = \frac{1}{2}(2\pi)^d \int \mathrm{d}^d k \, \tilde{\phi}(k) \tilde{\Delta}_{\Lambda}^{-1}(k) \tilde{\phi}(-k) + \mathcal{V}(\phi, \Lambda),$$

takes the form (omitting the term independent of ϕ)

$$\begin{aligned} \Lambda \frac{\mathrm{d}}{\mathrm{d}\Lambda} \mathcal{H}(\phi, \Lambda) = \frac{1}{2} \int \frac{\mathrm{d}^d k}{(2\pi)^d} \tilde{D}_{\Lambda}(k) \left[\frac{\delta^2 \mathcal{H}}{\delta \tilde{\phi}(k) \delta \tilde{\phi}(-k)} - \frac{\delta \mathcal{H}}{\delta \tilde{\phi}(k)} \frac{\delta \mathcal{H}}{\delta \tilde{\phi}(-k)} \right] \\ + \int \frac{\mathrm{d}^d k}{(2\pi)^d} \tilde{L}_{\Lambda}(k) \frac{\delta \mathcal{H}}{\delta \tilde{\phi}(k)} \tilde{\phi}(k) \end{aligned} \tag{64}$$

with (equation (56))

$$\tilde{L}_{\Lambda}(k) = \tilde{D}_{\Lambda}(k) / \tilde{\Delta}_{\Lambda}(k). \tag{65}$$

In equation (64), we have implicitly subtracted both from $\mathcal{H}(\phi, \Lambda)$ and from the equation their values at $\phi = 0$.

RGE: Standard form

After a field renormalization, required to be able to reach non-Gaussian fixed points, the RGE take the form (59) with $s = -\ln\Lambda$:

$$\begin{aligned}
&\Lambda \frac{\partial}{\partial \Lambda} \mathcal{H}(\phi, \Lambda) \\
&\quad = \frac{1}{2} \int \mathrm{d}^d x \, \mathrm{d}^d y \, D_\Lambda(x-y) \left[\frac{\delta^2 \mathcal{H}}{\delta\phi(x)\delta\phi(y)} - \frac{\delta \mathcal{H}}{\delta\phi(x)} \frac{\delta \mathcal{H}}{\delta\phi(y)} \right] \\
&\qquad + \int \mathrm{d}^d x \, \mathrm{d}^d y \, \phi(x) L_\Lambda(x-y) \frac{\delta \mathcal{H}}{\delta\phi(y)} + \frac{\eta}{2} \int \mathrm{d}^d x \, \phi(x) \frac{\delta \mathcal{H}}{\delta\phi(x)} \,. \qquad (66)
\end{aligned}$$

The function η is *a priori* arbitrary but with one restriction, it must depend on Λ only through $\mathcal{H}(\phi, \Lambda)$. It must be adjusted to ensure the existence of fixed points.

The latter equation does not have a stationary Markovian form since D_Λ and L_Λ depend explicitly on Λ:

$$L_\Lambda(x) = \frac{1}{(2\pi)^d} \int \mathrm{d}^d k \ \mathrm{e}^{ikx} \, \tilde{L}_\Lambda(k) = \Lambda^d L_1(x), \quad D_\Lambda(x) = \Lambda^{d-2} D_1(\Lambda x).$$

To eliminate this dependence, we perform a Gaussian renormalization of the form $\phi \mapsto \phi'$ with

$$\phi'(x) = \Lambda^{(2-d)/2} \phi(x/\Lambda).$$

In what follows, we omit the primes. Moreover, we introduce the dilatation parameter $\lambda = \Lambda_0/\Lambda$, which relates the initial scale Λ_0 to the running scale Λ and thus

$$\Lambda \frac{\mathrm{d}}{\mathrm{d}\Lambda} = -\lambda \frac{\mathrm{d}}{\mathrm{d}\lambda} \, .$$

Then, the RGE take a form consistent with the general RG flow equation:

$$\lambda \frac{\mathrm{d}}{\mathrm{d}\lambda} \mathcal{H}(\phi, \lambda) = \mathcal{T}\left[\mathcal{H}(\phi, \lambda)\right]$$

with

$$\begin{aligned}\mathcal{T}\left[\mathcal{H}\right] = & -\frac{1}{2} \int \mathrm{d}^d x \, \mathrm{d}^d y \, D(x-y) \left[\frac{\delta^2 \mathcal{H}}{\delta\phi(x)\delta\phi(y)} - \frac{\delta \mathcal{H}}{\delta\phi(x)} \frac{\delta \mathcal{H}}{\delta\phi(y)}\right] \\ & - \int \mathrm{d}^d x \, \frac{\delta \mathcal{H}}{\delta\phi(x)} \left[\frac{1}{2}(d-2+\eta) + \sum_\mu x^\mu \frac{\partial}{\partial x^\mu}\right] \phi(x) \\ & - \int \mathrm{d}^d x \, \mathrm{d}^d y \, L(x-y) \frac{\delta \mathcal{H}}{\delta\phi(x)} \phi(y). \end{aligned} \tag{67}$$

This form is more suitable for looking for fixed points. At a fixed point, the right hand side vanishes for a properly chosen renormalization of the field ϕ, which determines the value of the exponent η.

Expansion in powers of the field: RGE in component form

If one expands $\mathcal{H}(\phi, \lambda)$ (and then equation (67)) in powers of $\phi(x)$,

$$\mathcal{H}(\phi, \lambda) = \sum_{n=0} \frac{1}{n!} \int \prod_i \mathrm{d}^d x_i \, \phi(x_i) \mathcal{H}^{(n)}(x_1, \ldots, x_n; \lambda),$$

one derives equations for the components. For $n \neq 2$ ($D_1 \equiv D$),

$$\begin{aligned} \lambda \frac{\mathrm{d}}{\mathrm{d}\lambda} \mathcal{H}^{(n)}(x_i; \lambda) &= \left(\frac{1}{2} n(d + 2 - \eta) + \sum_{j,\mu} x_j^\mu \frac{\partial}{\partial x_j^\mu} \right) \mathcal{H}^{(n)}(x_i; \lambda) \\ &- \frac{1}{2} \int \mathrm{d}^d x \, \mathrm{d}^d y \, D(x - y) \Big[\mathcal{H}^{(n+2)}(x_1, x_2, \ldots, x_n, x, y; \lambda) \\ &- \sum_I \mathcal{H}^{(l+1)}(x_{i_1}, \ldots, x_{i_l}, x; \lambda) \mathcal{H}^{(n-l+1)}(x_{i_{l+1}}, \ldots, x_{i_n}, y; \lambda) \Big], \end{aligned}$$

where the set $I \equiv \{i_1, i_2, \ldots, i_l\}$ describes all distinct subsets of $\{1, 2, \ldots, n\}$.

In the Fourier representation, the equations take the form

$$\lambda \frac{\mathrm{d}}{\mathrm{d}\lambda} \tilde{\mathcal{H}}^{(n)}(p_i;\lambda) = \left(d - \frac{1}{2} n(d-2+\eta) - \sum_{j,\mu} p_j^\mu \frac{\partial}{\partial p_j^\mu} \right) \tilde{\mathcal{H}}^{(n)}(p_i;\lambda)$$
$$- \frac{1}{2} \int \frac{\mathrm{d}^d k}{(2\pi)^d} \tilde{D}(k) \tilde{\mathcal{H}}^{(n+2)}(p_1, p_2, \ldots, p_n, k, -k; \lambda)$$
$$+ \frac{1}{2} \sum_I \tilde{D}(p_0) \tilde{\mathcal{H}}^{(l+1)}(p_{i_1}, \ldots, p_{i_l}, p_0; \lambda) \tilde{\mathcal{H}}^{(n-l+1)}(p_{i_{l+1}}, \ldots, p_{i_n}, -p_0), \quad (68)$$

where the momentum p_0 is determined by total momentum conservation.

For $n = 2$, one finds an additional term in the equation:

$$\lambda \frac{\mathrm{d}}{\mathrm{d}\lambda} \mathcal{H}^{(2)}(x_1;\lambda) = \left(d + 2 - \eta + \sum_\mu x_1^\mu \frac{\partial}{\partial x_1^\mu} \right) \mathcal{H}^{(2)}(x_1;\lambda)$$
$$- \frac{1}{2} \int \mathrm{d}^d x \, \mathrm{d}^d y \, D(x-y) \left[\mathcal{H}^{(4)}(x_1, 0, , x, y; \lambda) - 2\mathcal{H}^{2)}(x - x_1;\lambda)\mathcal{H}^{(2)}(y;\lambda) \right]$$
$$- 2 \int \mathrm{d}^d y \, L(x_1 - y) \mathcal{H}^{(2)}(y;\lambda).$$

One verifies explicitly that, except in the Gaussian example, all functions $\mathcal{H}^{(n)}$ are coupled. In the Fourier representation,

$$\lambda \frac{\mathrm{d}}{\mathrm{d}\lambda} \tilde{\mathcal{H}}^{(2)}(p;\lambda) = \left(2 - \eta - \sum_\mu p^\mu \frac{\partial}{\partial p^\mu} \right) \tilde{\mathcal{H}}^{(2)}(p;\lambda) - 2\tilde{L}(p)\tilde{\mathcal{H}}^{(2)}(p;\lambda)$$
$$- \frac{1}{2} \int \frac{\mathrm{d}^d k}{(2\pi)^d} \tilde{D}(k) \tilde{\mathcal{H}}^{(4)}(p, -p, k, -k; \lambda) + \tilde{D}(p) \left(\tilde{\mathcal{H}}^{(2)}(p;\lambda) \right)^2 . \quad (69)$$

Perturbative solution

The flow equations in any of the different forms, can be solved perturbatively. One first specifies the form of the Hamiltonian at the initial scale $\lambda = 1$, for instance,

$$\mathcal{H}(\phi) = \mathcal{H}_{\mathrm{G}}(\phi) + \frac{g}{4!} \int \mathrm{d}^d x \, \phi^4(x), \quad g \geq 0 \, .$$

In the finite form (58) of the RGE, one can then expand the field integral in powers of the constant g. Alternatively, in the differential form, one first expands in powers of the field ϕ. This leads to the infinite set of coupled integro-differential equations (68,69) that one can integrate perturbatively with the Ansatz that the terms in $\mathcal{H}(\phi;\lambda)$ quadratic and quartic in ϕ are of order g and the general term of degree $2n$ of order g^{n-1}.

It is also possible to further expand the equations in powers of $\varepsilon = 4 - d$ and to look for fixed points. The results of the perturbative RG are then recovered.

Fixed points and local RG flow

Once a fixed point $\mathcal{H}_*$ has been determined, one can expand the FRG equation for $\mathcal{H}(\lambda)$ in the vicinity of the fixed point. We set

$$\mathcal{H}(\lambda) = \mathcal{H}_* + E(\lambda).$$

Then $E(\lambda)$ satisfies the linearized RGE

$$\lambda \frac{\mathrm{d}}{\mathrm{d}\lambda} E(\lambda) = \mathcal{L}_* E(\lambda),$$

where the linear operator $\mathcal{L}_*$, after an integration by parts, takes the form

$$\begin{aligned} \mathcal{L}_* = & \int \mathrm{d}^d x\, \phi(x) \left[\frac{1}{2}(d+2-\eta) + \sum_\mu x^\mu \frac{\partial}{\partial x^\mu} \right] \frac{\delta}{\delta\phi(x)} \\ & + \int \mathrm{d}^d x\, \mathrm{d}^d y\, D(x-y) \left[-\frac{1}{2} \frac{\delta^2}{\delta\phi(x)\delta\phi(y)} + \frac{\delta\mathcal{H}_*}{\delta\phi(x)} \frac{\delta}{\delta\phi(y)} \right] \\ & - \int \mathrm{d}^d x\, \mathrm{d}^d y\, L(x-y) \phi(x) \frac{\delta}{\delta\phi(y)}. \end{aligned} \tag{70}$$

We denote by ℓ the eigenvalues and $E_\ell \equiv E_\ell(\lambda = 1)$ the eigenvectors of $\mathcal{L}_*$:

$$\mathcal{L}_* E_\ell = \ell E_\ell \,, \tag{71}$$

and thus

$$E_\ell(\lambda) = \lambda^\ell E_\ell(1).$$

Equation (71) can be written more explicitly in terms of the components $\tilde{E}_\ell^{(n)}(p_i)$ of E_ℓ as

$$\begin{aligned} &\ell \tilde{E}_\ell^{(n)}(p_i) \\ &\quad = \left(d - \frac{1}{2} n (d - 2 + \eta) - \sum_j \tilde{D}(p_j) \tilde{\Delta}^{-1}(p_j) - \sum_{j,\mu} p_j^\mu \frac{\partial}{\partial p_j^\mu} \right) \tilde{E}_\ell^{(n)}(p_i) \\ &\qquad - \frac{1}{2} \int \frac{\mathrm{d}^d k}{(2\pi)^d} \tilde{D}(k) \tilde{E}_\ell^{(n+2)}(p_1, p_2, \ldots, p_n, k, -k) \\ &\qquad + \sum_I \tilde{D}(p_0) \tilde{E}_\ell^{(l+1)}(p_{i_1}, \ldots, p_{i_l}, p_0) \tilde{\mathcal{H}}_*^{(n-l+1)}(p_{i_{l+1}}, \ldots, p_{i_n}, -p_0). \end{aligned} \tag{72}$$

Gaussian fixed point

At the Gaussian fixed point, the Hamiltonian is quadratic and $\eta = 0$. The local flow at the fixed point is then governed by the operator

$$\mathcal{L}_* = \int \mathrm{d}^d x \, \phi(x) \left(\frac{1}{2}(d+2) + \sum_\mu x^\mu \frac{\partial}{\partial x^\mu} \right) \frac{\delta}{\delta \phi(x)}$$
$$- \frac{1}{2} \int \mathrm{d}^d x \, \mathrm{d}^d y \, D(x-y) \frac{\delta^2}{\delta \phi(x) \delta \phi(y)} \, .$$

The eigenvectors are obtained by choosing $\mathcal{H}^{(n)}$ vanishing for all n larger than some value N. The coefficient of the term of degree N in ϕ then satisfies the homogeneous equation

$$\ell \tilde{E}_\ell^{(N)}(p_i) = \left(d - \frac{1}{2} N(d-2) - \sum_{j,\mu} p_j^\mu \frac{\partial}{\partial p_j^\mu} \right) \tilde{E}_\ell^{(N)}(p_i).$$

This is an eigenvalue equation identical to the one obtained in the perturbative RG. The solutions are homogeneous polynomials in the momenta.

If r is the degree in the variables p_i, the eigenvalue is given by

$$\ell = d - \tfrac{1}{2} N(d-2) - r\,.$$

The other coefficients $\mathcal{H}^{(n)}$, $n < N$, are then entirely determined by the equations

$$\left(\frac{1}{2}(N-n)(d-2) + r - \sum_{j,\mu} p_j^\mu \frac{\partial}{\partial p_j^\mu}\right) \tilde{E}_\ell^{(n)}(p_i)$$
$$= \frac{1}{2}\int \frac{\mathrm{d}^d k}{(2\pi)^d} \tilde{D}(k) \tilde{E}_\ell^{(n+2)}(p_1, p_2, \ldots, p_n, k, -k).$$

One might be surprised by the occurrence of these additional terms. In fact, one can verify that if one sets

$$E(\phi)=\exp\left[-\frac{1}{2}\int \mathrm{d}^d x\,\mathrm{d}^d y\,\Delta(x-y)\frac{\delta^2}{\delta\phi(x)\delta\phi(y)}\right]\Omega(\phi),$$

the functional $\Omega(\phi)$ satisfies the simpler eigenvalue equation

$$\ell\,\Omega_\ell(\phi)=\int \mathrm{d}^d x\,\phi(x)\left[\frac{1}{2}(d+2)+\sum_\mu x^\mu\frac{\partial}{\partial x^\mu}\right]\frac{\delta}{\delta\phi(x)}\Omega_\ell(\phi),$$

whose solutions are the simple monomials $O_{n,k}(\phi,x)$ found in the perturbative analysis of the stability of the Gaussian fixed point. For example, for

$$\Omega_\ell(\phi)=\int \mathrm{d}^d x\,\phi^m(x),$$

after some algebra, one verifies

$$\ell=d-m(d-2)/2\,.$$

The linear operator that transforms $\Omega(\phi)$ into $E(\phi)$,

$$\exp\left[-\frac{1}{2}\int \mathrm{d}^d x\, \mathrm{d}^d y\, \Delta(x-y)\frac{\delta^2}{\delta\phi(x)\delta\phi(y)}\right],$$

replaces all monomials in ϕ that contribute to Ω by their normal products.

We recall that the normal product $E^{(N)}(\phi)$ of a monomial of degree N in ϕ is a polynomial with the same term of order N and such that, for all $n < N$, the Gaussian correlation functions

$$\left\langle \prod_{i=1}^{n} \phi(x_i) E^{(N)}(\phi) \right\rangle$$

with the measure $\mathrm{e}^{-\mathcal{H}_*}$ vanish. Let us point out that the definition of normal products depends explicitly on the choice of a Gaussian measure.

Beyond the Gaussian model: perturbative solution

In dimension 4 (and then in $d = 4 - \varepsilon$), a non-trivial theory can be defined and parametrized, for example, in terms of $g(\lambda)$, the value of $\tilde{\mathcal{H}}^{(4)}(p_i, \lambda)$ at $p_1 = p_2 = p_3 = p_4 = 0$:

$$g(\lambda) \equiv \tilde{\mathcal{H}}^{(4)}(p_i = 0, \lambda). \tag{73}$$

One then introduces the function $\beta(g)$ defined by

$$\lambda \frac{\mathrm{d}g}{\mathrm{d}\lambda} = -\beta\big(g(\lambda)\big). \tag{74}$$

This allows substituting in the left hand side of the flow equation

$$\lambda \frac{\mathrm{d}}{\mathrm{d}\lambda} = -\beta(g) \frac{\mathrm{d}}{\mathrm{d}g}.$$

One then solves the flow equations as power series in g, with appropriate boundary conditions.

All other interactions are then determined as well as series in g, under the assumption that they are at least of order g^2. They become implicit functions of λ through $g(\lambda)$. One suppresses in this way all corrections due to irrelevant operators, keeping only the contributions due to the marginal operator. The Hamiltonian flow is reduced to the flow of $g(\lambda)$, like in the perturbative renormalization group, but the fixed point Hamiltonian is much more complicate, since all subleading corrections to the leading behaviour are suppressed.

Finally, the function $\eta(g)$ is determined by the condition

$$\frac{\partial}{\partial p^2}\tilde{\mathcal{H}}^{(2)}(p;g)\bigg|_{p=0}=1 \;\Rightarrow\; \frac{\partial}{\partial p^2}\tilde{\mathcal{V}}^{(2)}(p;g)\bigg|_{p=0}=0\,, \tag{75}$$

which suppresses the redundant operator that corresponds to a change of normalization of the field.

The two conditions (73), (75) replace the renormalization conditions of the usual renormalization theory.

Remarks

(i) In practice, various non-perturbative approximation scheme have been devised, reducing the functional RG equations to partial differential equations. Quite interesting results have then been obtained. The main problem is that none of these approximation scheme has a systematic character, or when a systematic method is claimed, the next approximation is out of reach.

(ii) Finally, since the effective interaction $\mathcal{V}(\phi)$ is the generating functional of some kind of connected Feynman diagrams, a further simplification of the FRG equations is obtained by expressing them in terms of the Legendre transform of $\mathcal{V}(\phi)$, which involves only one-line irreducible diagrams.